Physics 613 Lecture 18 April 8, 2014

Renormalizability

We saw in our oversimplified ABC theory that the only divergence was in the one-loop self energy. In general a loop will have a number of propagators, each of which gives, for scalars, two powers of momentum in the denominator, so if there are more than two, the $\int d^4k$ will not diverge. Also, as the vertex has only three propagators, such a loop cannot be part of an overlapping divergence, except for the irrelevant vacuum bubble in Fig. 6.3a. But the Dirac propagator has only one power of momentum in the denominator, so there may be, and are, divergences also in loops with three or four propagators.

One way to understand whether we can have divergences in a theory is to observe that the invariant amplitude with n external particles is supposed to have dimension $4-n$ (energy^{4-n}), as one can see from the formula for the cross section 6.110 with 6.111. The Lagrangian density has dimension 4 (energy^4) (in four-dimensions) and the scalar and vector fields have dimension 1, and the Dirac fields dimension 3/2. This means the parameters that enter the lagrangian have definite dimensions, 1 for m, 1 for g of any ϕ^3 coupling, 0 for e of $e\bar{\psi}\gamma^\mu\psi A_\mu$ and for λ of $\lambda\phi^4$. Note that all of these have non-negative dimensions. Any counterterms in the Lagrangian will be a function of the coupling constants or parameters of the Lagrangian, multiplying fields and the total dimension must be four. As all the fields have positive dimensions, if all parameters have non-negative dimension, only a few combinations of fields can appear.

Consider a Feynman diagram with B external bosons, F external fermions, N_B internal boson propagators, N_F internal fermion propagators, N_g ϕ^3 vertices, N_λ ϕ^4 vertices, N_m mass insertions, N_ψ couplings of $\bar{\psi}\Gamma\psi$ to bosons, and N_L loop integrals. The "superficial degree of divergence" is just power counting in the internal momenta, $4N_L - 2N_B - N_F$, which will indicate an ultraviolet divergence if it is ≥ 0 (a logarithmic divergence if 0).

But the number of ends of propagators plus external particles of each type must correspond to the number of attachment points on the vertices for that type, so

$$B + 2N_B = 2N_m + 3N_g + 4N_\lambda + N_\psi \tag{1}$$

$$F + 2N_F = 2N_\psi \tag{2}$$

There are 4D-momenta for each propagator, with one 4D constraint for each vertex, except for one giving overall momentum conservation, so the number of unconstrained momenta (the number of loops) is

$$N_L = N_B + N_F - N_m - N_g - N_\lambda - N_\psi + 1.$$

Thus the superficial degree of divergence is

$$\begin{aligned} 4N_L - 2N_B - N_F &= 4 + 4N_B + 4N_F - 4N_m - 4N_g - 4N_\lambda - 4N_\psi - 2N_B - N_F \\ &= 4 + (2N_m + 3N_g + 4N_\lambda + N_\psi - B) + \frac{3}{2}(2N_\psi - F) \\ &\qquad -4N_m - 4N_g - 4N_\lambda - 4N_\psi \\ &= 4 - 2N_m - N_g - B - \frac{3}{2}F \end{aligned}$$

Notice that this corresponds to just $4 - \sum$ (dimensions of constants plus external fields). Thus we can only have divergences with four or fewer external bosons, or two fermions and up to one boson.

Now if we had vertices with higher powers of the fields, they would enter the ends-counting equations ([1] and [2]) with higher coefficients and would give positive contributions to the degree of divergence, so there would be divergent diagrams with arbitrary numbers of external particles. Each of these would need an arbitrary counterterm and the theory would have no predictive power. Such theories are called non-renormalizable. But as long as all the coupling constants have non-negative dimensions, the counterterms are limited to those listed, 2, 3, and 4 boson vertices, and fermion-antifermion-boson and fermion mass counterterms. So these are renormalizable theories. [Note: this depends on our living in 4 dimensions. In three space-time dimensions ϕ^6 would be renormalizable.]

In the discussion above, we assumed the vector propagator behaved like momentum^{-2}, which is OK if we can use $-g_{\mu\nu}/q^2$. That depended on gauge invariance and the Ward identity. We might ask what would happen if we had a field theory with massive vector particles. The field B_μ represents a free particle obeying $(\Box + m^2)B^\mu = 0$ with a constraint that says in its rest frame, B_μ is a spin=1 with no timelike component. That is, $\partial_\mu B^\mu = 0$. Thus the equation of motion is really $\partial^\mu \partial_\mu B^\nu - \partial^\nu \partial_\mu B^\mu + m^2 B^\nu = 0$. In taking ∂_ν of this equation, the first two terms cancel, and this implies $\partial_\mu B^\mu = 0$, but having established that, the second term vanishes and we get the Klein-Gordon equation. In momentum space, this equation is $[-(k^2 - m^2)g_{\mu\nu} + k_\mu k_\nu]\tilde{B}^\nu = 0$.

The propagator needs to be i times the inverse of $-(k^2-m^2)g_{\mu\nu}+k_\mu k_\nu$, which is

$$D_F^{\mu\nu} = \frac{-g^{\mu\nu} + k^\mu k^\nu/m^2}{k^2 - m^2}.$$

But notice that this propagator does not behave like k^{-2} in the ultraviolet (at large k), and does not help Feynman diagrams to converge. Without the $-2N_B$ the bosonic propagators gave in the degree-of-divergence formula, there is no limit on how many external particles, as well as on coupling constants and loops, that can diverge. So a field theory which starts off with massive spin-1 particles is non-renormalizable. We will see later that there is a way around this, by starting with zero mass vector particles but introducing spontaneous symmetry breaking.

Quantum Electrodynamics

The quantum field theory that really established its pertinence in real physics was QED, Quantum Electrodynamics, with a lagrangian

$$\mathcal{L} = \bar\psi_0(i \not\partial - m_0)\psi_0 - e_0\bar\psi_0\gamma^\mu\psi_0 A_{0\,\mu} - \frac{1}{4}F_{0\,\mu\nu}F_0^{\mu\nu} - \frac{1}{2\xi_0}\left(\partial_\mu A_0^\mu\right)^2,$$

where we have subscripted all the parameters and fields with 0 to indicate that they are not the renormalized (or "physical") values. To get a theory in which each new order in perturbation theory makes a small, rather than infinite, change from the previous order, we reexpress these "bare" fields and parameters in terms of the renormalized ones, which we now write without a $_{\rm ph}$ subscript. The renormalized fields are

$$\psi = Z_2^{-1/2}\psi_0, \qquad A^\mu = Z_3^{-1/2}A_0^\mu$$

and the lagrangian becomes $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$, with

$$\mathcal{L}_0 \;=\; \bar\psi(i \not\partial - m)\psi - -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2, \qquad (3)$$

$$\mathcal{L}_I \;=\; -e\bar\psi\gamma^\mu\psi A_\mu \qquad (4)$$

$$+i(Z_2-1)\bar\psi \not\partial\psi - \delta m\bar\psi\psi - \frac{1}{4}(Z_3-1)F_{\mu\nu}F^{\mu\nu} - (Z_1-1)e\bar\psi\gamma^\mu\psi A_\mu. \qquad (5)$$

We now do the interaction picture using (3) as the bare lagrangian and now have interactions of the original type (4) and counterterms (5). We need an agreement on how the parameters of the lagrangian are split into "physical" ones and counterterms, and this will be by imposing renormalization conditions.

One of the renormalization conditions is that the "physical mass" is indeed what one measures for a single particle. When we calcuate the electron self-energy to a new order n, we use the physics mass and the physical charge, and then determine the new contribution $\mathcal{O}(e^n)$ to Z_2 and δm to keep the pole and residue in $\tilde{D}'_F$ at the physical mass with weight 1. One must also calculate the renormalization of the photon propagator, giving Z_3. It is the magic of gauge invariance that the photon propagator does not develop a mass shift, which has to do with the fact (assumption?) that there is no massless particle with the photon's quantum numbers. This will prove a crucial proviso when we consider gauge fields with spontaneously broken symmetries. It preserves the Ward identities. Finally, there is the renormalization of the fermion vertex, and Z_1, fixed to keep the physical charge of the electron at its measured value, with $Z_1 e = e_0 Z_2 \sqrt{Z_3}$.

Notice that for the electron derivative terms and the photon coupling to combine into the covariant derivative, it is necessary that $Z_1 = Z_2$, and in fact this is also due to the Ward identity.

The actual calculation of these one-loop graphs in QED is complicated by several issues. One is the masslessness of the photon, which gives rise to infrared divergences. Because a photon can carry off arbitrarily little energy and momentum, any scattering of a charged particle has zero probability of not emitting any photons. But our S matrix elements were defined in terms of precisely specified final states, and this leads to divergences in calculating the amplitude for that. It can be understood by placing a lower limit on the energies of the photons guaranteed not to be present in the final state (a kind of inclusive cross section). The other complications have to due with assuring the Ward identity is preserved. I am not going to discuss these issued further — you have seen it in 615 or may do so in the future. Instead, we will begin our discussion on non-Abelian symmetries.

Non-Abelian Symmetry

In the first two lectures this term, we discussed how crucial symmetries are in high energy theory. We have discussed both global symmetries, in which a symmetry transformation fixed by a finite number of parameters is applied to the fields at all space-time points, and local symmetry, in which the parameters are allowed to vary from point to point in space-time. Examples of global symmetry include both Poincaré invariance and isotopic spin or SU(3) flavor symmetries. Thus far the only local symmetry we have encountered is the phase invariance of charge fields coupled by convariant derivatives to electromagnetism. Phase transformations, $\psi \to e^{i\theta}\psi$ form a very simple group, translations modulo 2π or equivalently, rotations in a plane, $SO(2)$. But we know from rotations, isorotations, and flavor SU(3) that global symmetries can be considerably bigger and more complicated. Especially important is that the elements of the group do not commute — for example, the effect of two rotations about different axes depend on the order the two rotations are performed. Groups for which all elements commute are called Abelian, otherwise they are non-Abelian.

Groups can be *finite*, that is, have a finite number of elements, like the group of rotations that leave a cube invariant, which has 24 elements, or, if we include reflections, 48 elements. Or it can be infinite. If infinite, it can be discrete, such as the group of addition by integers, or it can be continuous, with real parameters, as in addition by reals or unrestricted rotations. In the case of continuous groups, one can generally[1] specify all that one needs in terms of the infinitesimal generators, as we do for rotations using the angular momenta L_x, L_y and L_z, in terms of which an arbitrary rotation can be written $R = e^{i\vec{\omega}\cdot\vec{L}}$, with $\vec{\omega}$ a 3-vector of real numbers.

For a global symmetry, if we have a physically allowed state of the system, and we apply a symmetry group transformation, the resultant state must also be a physically allowable state. That means states of the system form *representations* of the group. We are used to this from atomic states in quantum mechanics, where the three $n = 2, \ell = 1$ states of a hydrogen atom (ignoring spin) transform into each other under rotations. Given a particular state, not every state of the system is a group transformation of that state, of course. If we start with $|n = 2, \ell = 1, m = 0\rangle$ we can get to any state with $n = 2, \ell = 1$, which is a three-dimensional space, but not to any of the states with different n or ℓ. We say that this three-dimensional

[1] Not quite. For example, the group of rotations SO(3) and the group of special unitary matrices in two dimensions, SU(2), have the same generators, with the same commutation relations. But a rotation through 2π returns one to the identity, while $e^{2\pi i L_z}$ in SU(2) gives $-\mathbb{1}$. Perhaps more importantly, SO(2) and addition of reals each have a single generator, but the parameters multiplying it take on a finite domain for SO(2), but infinite for addition of reals.

space is an irreducible representation of SO(3). The way the states within a representation transform into each other is completely specified by the representation, so one doesn't need to know the detailed physical state. We saw in Lecture 2 how this enabled us to make statements of the scattering amplitudes of various pions off nucleons without knowing any details of the hadronic interaction, other than that when the particles are far apart they transform independently.

In that discussion, we made use of the fact that a state which consists of two independent parts, like the two particles well before collision, transforms like the product of the way the parts transform under finite transformations. The direct product of two irreducible representations can be decomposed into a sum of irreducible representations, and the way each state in the combined representation depends on the component states is given by what physicists call Clebsch-Gordon coefficients[2]. As the Hamiltonian or Lagrange density should be invariant under the symmetry, a term which is a product of fields must be a combination which is the identity representation, $M(g) = 1$ for all $g \in G$. We have already seen this for Lorentz invariance (and of course spin) as $\bar{\psi}\gamma^\mu\psi A^\nu g_{\mu\nu}$ is invariant because the $(\gamma^0\gamma^\mu)_{ab}$ are essentially a Clebsch-Gordon coefficients for combining a Dirac representation ψ_b and its conjugate $\bar{\psi}_a$ into a contravariant vector representation, and $g_{\mu\nu}$ are Clebsch-Gordon coefficients for combining two contravariant vector representations into the identity.

For a Lie group the group transformations are exponentials of symmetry *generators*, and so for the direct product of representations, the generators act as the direct sum. If you saw the derivation of Clebsch-Gordon coefficients for SO(3) in quantum mechanics class, you used this fact.

SU(3)$_{\textbf{flavor}}$

Let us return to the approximate symmetry that we could have if the three lighter quarks, u, d, and s are considered to be components of a complex 3-vector in flavor space, and the theory is as symmetric as possible. That would mean that the three quark fields $q_j(x) = u(x), d(x), s(x)$ could be replaced by

[2] Some mathematicians disagree — they say the Clebsch Gordon coefficients are the number of times a representation J occurs in the direct product $J_1 \otimes J_2$, so for SU(2) it is always either 0 or 1, but for larger groups a given representation may occur more than once, *e.g.* for SU(3) $\underline{\mathbf{8}} \otimes \underline{\mathbf{8}} = \underline{\mathbf{27}} + \underline{\mathbf{10}} + \underline{\bar{\mathbf{10}}} + \underline{\mathbf{8}} + \underline{\mathbf{8}} + \underline{\mathbf{1}}$. But the physicist's definition has taken over, at least at Wikipedia.

$q'_j = \sum_k W_{jk} q_k$ and the lagrangian would be unchanged. But there is surely a term $\bar{q}_j \gamma^\mu \partial_\mu q_j$ in the lagrangian, which means that, assuming W is global with no x^μ dependence, and is a Lorentz scalar, that $W^\dagger W = 1\!\!\mathrm{I}$, so W must be a unitary 3×3 complex matrix. So we might have a U(3) symmetry of our quark theory.

Actually, the group U(3) can be split into SU(3), which are unitary matrices with determinant 1, and U(1)=SO(2) which consists of $e^{i\theta}$ times the 3×3 unit matrix. As the latter matrices commute with all the SU(3) matrices, the group U(3) can be treated as two commuting, independent groups. Each symmetry, of course, generates a conserved charge, and the U(1) charge is baryon number B.

Any unitary matrix can be written as $U = e^{iH}$ with a hermitean matrix H, and $\det U = \exp(i \operatorname{Tr} H)$, so the generators of SU(3) are traceless hermitean 3×3 matrices. There are thus 8 independent generators (if you like, 3 complex numbers above the diagonal, and two for the diagonal with the tracelessness constraint). Thus the generators of SU(3) form an 8 dimensional vector space called the Lie algebra. We may define a standard basis λ_a of these, analogous to the three Pauli spin matrices σ_j. In fact, the first three λ's are the Pauli matrices with zeros in the last row and column. The most important other one is $\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. The generators are taken to be $T_j = \lambda_j/2$, and satisfy $\operatorname{Tr}(T_a T_b) = \delta_{ab}/2$, with $[T_a, T_b] = i f_{abc} T_c$ defining the *structure constants* f_{abc} of the Lie algebra.

There are 8 conserved quantities from the SU(3) symmetry, but as they do not commute, only two can be simultaneously diagonalized. This is the analog of L_z for SO(3). For SU(3), these are T_3 and T_8, but we usually use hypercharge $Y = \frac{1}{\sqrt{3}} \lambda_8$, so the u and d quarks have $Y = 1/3$, and the s quark $Y = -2/3$. $S = Y - B$ is the strangeness.

The quarks form a three dimensional representation known as $\underline{\mathbf{3}}$, and the antiquarks form a *different* three dimensional representation $\underline{\bar{\mathbf{3}}}$. You can tell they are not the same because the hypercharges are reversed, which is different from what happens for SU(2), where the antidoublet is equivalent to the doublet. So $\bar{q}$ transforms as the antiquark, and the combination $(\bar{q})_j (\lambda_a)_{jk} q_k$ transforms like an octet, that is, just like the generators of the group. If we have vector bosons which also transform like the generators, $-ig_s \bar{q}_j (\lambda_a)_{jk} \gamma_\mu q_k A_a^\mu$ is invariant under flavor SU(3). But this is not what we really have, because the gluons A_a^μ transform under color SU(3), and are flavor singlets. But, as we shall see, the quarks are not only a triplet under flavor, but also a triplet under color, and the above term should really be replaced by

$$-ig_s \sum_f \bar{q}_j^{(f)} (\lambda_a)_{jk} \gamma_\mu q_k^{(f)} A_a^\mu$$

where $q_k^{(f)}$ is a quark of flavor $f = u, d, s$ and color $j, k =$ blue, green, or red.