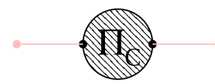
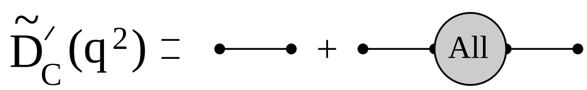
Physics 613 Lecture 17 April 3, 2014

Renormalization

Last time we began our discussion of feynman graphs containing loops, and therefore some propagators with momentum not fixed by the external momenta and momentum conservation. In particular, we found self-energy corrections could occur on propagators or external lines.

We defined $\Pi_C(q^2)$ to be the sum of all one-particle-irreducible (1PI) graphs that could fit between two bare propagators $\tilde{D}_F(q, m_C)$, and then we saw that the sum of *all* connected 2-point graphs gave





$$= \frac{i}{q^2 - m_c^2 - \Pi_c(q^2)},$$

and that the physical mass of a lone C particle is the value $m_{\rm ph}$ for which $m_{\rm ph}^2 = m_C^2 + \Pi_C(m_{\rm ph}^2)$. For values of q^2 near $m_{\rm ph}^2$, we can expand

$$\Pi_C(q^2) = \Pi_C(m_{\rm ph}^2) + (q^2 - m_{\rm ph}^2) \left.\frac{d\Pi_C}{dq^2}\right|_{q^2=m_{\rm ph}^2}$$

$$\text{so} \quad \tilde{D}'_C(q^2) \approx \frac{i}{q^2 - m_C^2 - \Pi_C(m_{\rm ph}^2) - (q^2 - m_{\rm ph}^2) \left.\frac{d\Pi_C}{dq^2}\right|_{q^2=m_{\rm ph}^2}}$$

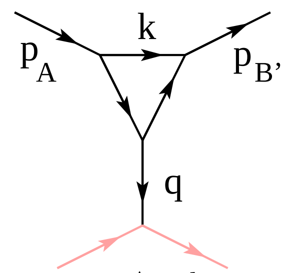
$$= \frac{i}{\left(q^2 - m_{\rm ph}^2\right)\left(1 - \left.\frac{d\Pi_C}{dq^2}\right|_{q^2=m_{\rm ph}^2}\right)} = \frac{iZ_C}{q^2 - m_{\rm ph}^2},$$

where

$$Z_C^{-1} = 1 - \left.\frac{d\Pi_C}{dq^2}\right|_{q^2=m_{\rm ph}^2}.$$

Z_C is known as the *field-strength renormalization*.

Vertex Corrections



Another diagram which enters the $\mathcal{O}(g^4)$ calculation of AB scattering is the vertex correction diagram. This will give a contribution of the form $-ig\frac{i}{q^2-m_C^2}\left(-igG^{[2]}(p_A,p'_B)\right)$ where $G(p_A,p'_B)$ can be viewed as the sum of all graphs connectable to three propagators, and thus a generalization of the vertex, with $G^{[0]}=1$, and our diagram the $\mathcal{O}(g^2)$ contribution[1] to an infinite power series expansion. As these insertions will occur whereever a vertex can, any attempt to measure the value of g will have these corrections included. This will be more interesting to discuss in a real process such as the charge in QED, but for now we will just comment that the *physical charge* will not be the value g in the Lagrangian but instead a corrected g_{ph}. The recognition that the physical values of the mass and coupling constants are modified from the values of the corresponding parameters in the Lagrangian is called *renormalization*. The unpleasant reality that this renormalization is infinite requires us to introduce artificial cutoffs to make the renormalization finite and well-determined. This is called *regularization*. Then, because we don't really care about or have any way of observing the parameters in the Lagrangian, if we can remove the cutoffs while leaving the physical parameters fixed, we have a well-defined theory.

Calculation of $\Pi^{[2]}$

Let us return to the evaluation of

$$\Pi^{[2]}(q^2) = -ig^2\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m_A^2+i\epsilon}\,\frac{i}{(q-k)^2-m_B^2+i\epsilon}.$$

The four-dimensional integral is complicated by the two denominators — were there only one denominator, we could use Lorentz invariance to eliminate the angular dependence. In fact, we would first deform the contour integral in k^0 from the real axis to the imaginary axis, taking it away from the poles, and, setting $k^0 = ik_E\ k^j = k_E^j$, we are now integrating Euclidean $d^4k_E^\mu$, and the denominator is $-((k_E^0)^2+\vec{k}_e^{\,2}+m^2) = -(k_E^2+m^2)$ which has no chance of vanishing (if $m\neq 0$). Also, the integrand would then by hyperspherically symmetric, so the angular integral would just give the volume of

[1]I would have called these $G^{[1]}=-ig$ and $G^{[3]}$ respectively, but, Oh well.

a 3-sphere, that is, the volume of the surface of a four dimensional ball of radius k, which is[2] $2\pi^2 k^3$. But with multiple denominators, the symmetry around $k = 0$ is disturbed by the $(q-k)^2$ term. For any loop that doesn't overlap with other loops, we have a single four-dimensional integral over momentum but a product of denominators with $(k-p_j)^2$ terms, which appears very difficult. But there is a trick, due to Feynman or Schwinger[3] which tells us[4] that

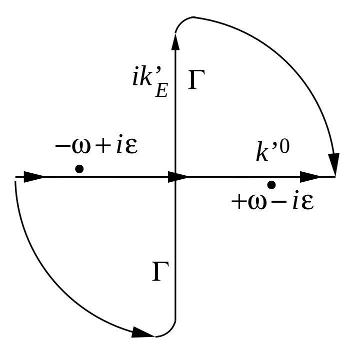
$$\prod_{i=1}^{N} \frac{1}{A_i} = \Gamma(N) \left(\prod_{i=1}^{N} \int_0^\infty d\alpha_i \right) \delta\left(1 - \sum_{i=1}^{N} \alpha_i\right) \left[\sum \alpha_i A_i\right]^{-N}.$$

We have two denominators, $N = 2$, and if we let $x = \alpha_2$, this reads $\dfrac{1}{AB} = \dfrac{dx}{[(1-x)A + xB]^2}$, and

$$\begin{aligned} -i\Pi^{[2]}(q^2) &= g^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(1-x)(k^2 - m_A^2 + i\epsilon) + x\{(q-k)^2 - m_B^2 + i\epsilon\}]^2} \\ &= g^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - 2xk\cdot q + x^2 - (1-x)m_A^2 - xm_B^2 + i\epsilon]^2} \\ &= g^2 \int_0^1 dx \int \frac{d^4k'}{(2\pi)^4} \frac{1}{(k'^2 - \Delta + i\epsilon)^2} \end{aligned}$$

where $k' = k - xq$ and $\Delta(q^2, x) = -x(1-x)q^2 - xm_B^2 - (1-x)m_A^2$.

Now, for a fixed value of x and $\vec{k}'$, the integral dk'^0 has poles at $dk'^0 = \pm\sqrt{(\vec{k}')^2 + \Delta - i\epsilon}$, and at least for spacelike q, these are just off the real axis in the same way as we had for the propagator in lecture 9. So we can deform the dk' contour so that it runs up the imaginary axis, call its imaginary part k'_E, and observe that the integral is now



The

$$-i\Pi^{[2]}(q^2) = ig^2 \int_0^1 dx \int \frac{d^4k'_E}{(2\pi)^4} \frac{1}{(k'^2_E + \Delta)^2}.$$

k'^2_E is now a Euclean four-vector squared, and we have no angular dependence,

[2]See http://www.physics.rutgers.edu/grad/615/lects/gammanSn.pdf.

[3]For a bit of petulance about this, see Schwinger, "Particles, Sources, and Fields", p. 338.

[4]See http://www.physics.rutgers.edu/grad/615/lects/schwingertrick.pdf.

so

$$-i\Pi^{[2]}(q^2) = ig^2 \int_0^1 dx \int_0^\infty \frac{2\pi^2 k^3\,dk}{(2\pi)^4} \frac{1}{(k^2+\Delta)^2} = i\frac{g^2}{8\pi^2} \int_0^1 dx \int_0^\infty \frac{k^3\,dk}{(k^2+\Delta)^2}.$$

In the book, this integral was done differently, and they get an apparently different answer as Eq. 10.50, which has $\int_0^\infty \frac{u^2 du}{(u^2+\Delta)^{3/2}}$ instead of $\int_0^\infty \frac{k^3\,dk}{(k^2+\Delta)^2}$. Whether they are the same is a dubious question, because neither integral is well defined. In fact, I have been negligent in throwing away the two quarter-circles at infinity, which would have added the constant $ig^2/16\pi^3$. The difference is independent of q^2, because if we differentiate the two expressions with respect to Δ (which is where the q^2 dependence enters), we get no contribution from the arcs at infinity, and indeed the two expressions give the same expression. In fact, the derivative can be explicitly evaluated[5],

$$\int_0^\infty du \frac{d}{d\Delta}\left(\frac{u^2}{(u^2+\Delta)^{3/2}}\right) = \int_0^\infty dk \frac{d}{d\Delta}\left(\frac{k^3}{(k^2+\Delta)^2}\right) = -\frac{1}{2\Delta},$$

which means that

$$\Pi^{[2]}(q^2) = C + \frac{g^2}{16\pi^2}\int_0^1 dx \ln(-x(1-x)q^2 + xm_B^2 + (1-x)m_A^2),$$

where C is independent of q^2, so although it is infinite, it is also irrelevant. The field-strength renormalization is

$$Z_C \approx 1 + \left.\frac{d\Pi_C^{[2]}}{dq^2}\right|_{q^2=m^2_{\rm ph,C}} = -\frac{g^2}{16\pi^2}\int_0^1 dx \frac{x(1-x)}{-x(1-x)m^2_{\rm ph,C} + xm_B^2 + (1-x)m_A^2}.$$

As the denominator takes on its minimum value for $x = \frac{1}{2} + \frac{m_A^2 - m_B^2}{2mC^2}$ at which it takes the value

$$-\frac{m_C^4 + m_A^4 + m_B^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}{4m_C^2}$$
$$= \frac{m_A + m_B + m_C}{4m_C^2}\,(m_A + m_B - m_C)\,(m_A - m_B + m_C)\,(-m_A + m_B + m_C)\,,$$

[5]This provides an instructive example of how one must be careful in manipulating ill-defined objects. If one looks at either of the integrals above, either mine or the book's, and scales the integration variable $k = \sqrt{\Delta}v$ or $u = \sqrt{\Delta}v$, one finds the integrals are independent of Δ, but here we see that it has a non-zero derivative.

which we see vanishes whenever and only whenever two of the masses add up to the third. So as long as each particle is stable against decay into the two others, the denominator does not vanish, the integral is finite, and the field-strength renormalization is finite (and order g^2).

Counterterms

Our ABC theory is considerably simpler than the actual field theories we need to discuss high energy theory. What we have seen here is due to its super-renormalizable nature. The only infinity si the displacement of the physical mass from the lagrangian mass. We are really interested in just renormalizable (not super-) theories. The distinction in general is that super-renormalizable theories have only a finite number of divergent Feynman graphs, not counting divergences within larger graphs. In ABC the only one is the self-energy graph $\Pi^{[2]}$, though of course that bubble could appear within any graph on one of its propagators. For renormalizable theories there are only a finite number of invariant matrices which diverge, but there can be infinitely many graphs which can contribute. More relevant to the present discussion, however, is that the field-strength and coupling constant renormalization are finite in ABC theory but not in QED and other really relevant theories. Each of these infinities will affect only the values of parameters in the lagrangian, and their relation to physical observables, so that if we regulate the theory, perhaps with a cutoff, and hold the physical observables fixed while we let the cutoff go away, the things that diverge are only the parameters in the lagrangian, and physical predictions are okay.

Still, it is disconcerting to be basing our calculations on perturbation theory around a free theory which differs infinitely from our observed values, and it would be better if we could systematically do a perturbation around a basis that was not too far off. To do that, we rewrite the division between the bare lagrangian $\hat{\mathcal{L}}_{0,A} = \frac{1}{2}\partial_\mu\hat{\phi}_A\partial^\mu\hat{\phi}_A - \frac{1}{2}m_{0,C}^2\hat{\phi}_A^2$ and $\hat{\mathcal{L}}_{0,B}$ and $\hat{\mathcal{L}}_{0,C}$ and the interaction piece $\hat{\mathcal{L}}_{\text{int}} = -g_0\hat{\phi}_A\hat{\phi}_B\hat{\phi}_C$. We first reexpress[6] it in terms of the renormalized field $\bar{\phi}_j(x) := Z_j^{-1/2}\hat{\phi}_j(x)$, the renormalized masses $\bar{m}_j$ and the renormalized coupling $\bar{g}$ and define the noninteracting lagrangian to be $\bar{\mathcal{L}}_{0,j} = \frac{1}{2}\partial_\mu\bar{\phi}_j\partial^\mu\bar{\phi}_j - \frac{1}{2}\bar{m}_{0,j}^2\bar{\phi}_j^2$ and throw the rest of $\hat{\mathcal{L}}_0$ into the interaction

[6]My notation differs from the book. I am calling the parameters in the original Lagrangian $m_{0,j}$ and g_0 and the field $\hat{\phi}$, but the renormalized quantities $\bar{m}_j$ $(= m_{\text{ph},j}$ in book), $\bar{g}$ (g_{ph} in book) and $\bar{\phi}_j$ $(= \hat{\phi}_{\text{ph},j}$ in book).

piece. This means that the bare Lagrangian fields will create particles with the correct weight, will have propagators that blow up at the right mass. Of course we have added to the interaction term

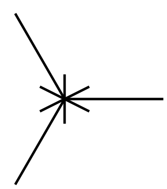
$$\hat{\mathcal{L}}_0 - \bar{\mathcal{L}}_0 = \sum_j \left\{ \frac{1}{2}\delta Z_j \partial_\mu \bar{\phi}_j \partial^\mu \bar{\phi}_j - \frac{1}{2}(\delta Z_j \bar{m}_{0,j}^2 + Z_j \delta m_j^2)\bar{\phi}_j^2 \right\},$$

which we can represent by . We also want to have the correct $\mathcal{O}(g)$ coupling constant $\bar{g}$, so we define a coupling constant renormalization constant Z_V with

$$Z_V \bar{g} = g_0 \sqrt{Z_A Z_B Z_C},$$

so the interaction term in the original lagrangian

$$\begin{aligned} &-g\hat{\phi}_A\hat{\phi}_B\hat{\phi}_C \\ &\quad = -\bar{g}\bar{\phi}_A\bar{\phi}_B\bar{\phi}_C \\ &\qquad -(Z_V - 1)\bar{g}\bar{\phi}_A\bar{\phi}_B\bar{\phi}_C, \end{aligned}$$



and the second term will be a new counterterm represented by this diagram.

Note that the division of $m_{0,j}$ and g into the counterterms and the lowest order piece is chosen by insisting, at each order in perturbation theory, that the particle mass and the three-point function, defined appropriately, be the observed values.