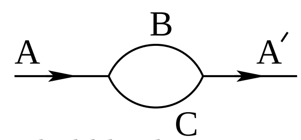
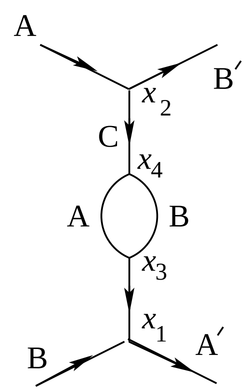
Physics 613 Lecture 16 April 1, 2014

Loops and Renormalization

In lectures 7 and 8 we saw that a perturbative expression for scattering amplitudes could be derived by using the interaction picture and expanding the S matrix, the exponential of $-i\int d^4x\hat{\mathcal{H}}_I(x)$, in powers of the coupling constants in $\mathcal{H}_I$, and we applied this to calculate the lowest order contributions to scattering and decay widths in ABC theory. In lectures 10 and 11 we discussed the Dirac and electromagnetic fields and in lectures 12-13 we gave the general rules for the perturbative expansion, but evaluated only the lowest order (in α) contributions to $e\mu$ scattering or e^+e^- annihilation. We did see that some contractions led to Feynman diagrams in which not all internal momenta were determined by momentum-conserving delta functions, such as the bubble on the A propagator in ABC theory, but we didn't pay attention there, as this did not affect the scattering amplitude. The same bubble, however, can appear on an internal propagator and then needs to be considered.



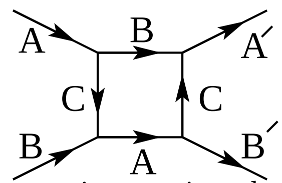
In AB scattering by C exchange, there can be a bubble on the C propagator, as shown here. Also if we turn this figure on its side, it is a correction to the C resonance contribution (Fig. 6.3d) which will correct for the infinity that diagram would give for the cross section when the incoming center-of-mass energy is m_C, or $s = m_C^2$.



The lowest order diagrams that contibuted to $A + B \to A' + B'$ scattering were order g^2, coming from the square of the $\int\mathcal{H}$ from the exponential. As $\mathcal{H}$ changes the numbers of each particle by ± 1, the cubic term in the exponential will not contibute to this particular scattering, so we need to go to fourth order, giving a contribution $\hat{S}^{(4)}$ of

$$\langle\vec{p}\,'_A\vec{p}\,'_B|\,\hat{S}^{(4)}\,|\vec{p}_A\vec{p}_B\rangle = \frac{(-i)^4}{4!}\sqrt{16E_AE'_AE_BE'_B}\int dx_1\int dx_2\int dx_3\int dx_4$$
$$\langle 0|\,\hat{a}_A(p'_A)\hat{a}_B(p'_B)\left(\prod_{j=1}^{4}\mathcal{H}(x_j)\right)a^\dagger_A(p_A)\hat{a}^\dagger_B(p_B)\,|0\rangle\,.$$

There are many contractions of the 12 $\hat{\phi}(x_j)$'s with the external particles and with each other, some equivalent and some giving different contributions. We will consider, today, those that give rise to the diagram shown above. For homework you will consider another contribution which gives a very different contribution. But for now, we see that the incoming B particle is annihilated in the same interaction that the A' is created. This could be any of the four factors of $\mathcal{H}$, but all will give the same contribution, so we will say it happened at x_1 and multiply by 4. Similarly the incoming A is annihilated in the same interaction that produces the B', and it could be any of the remaining three factors of H, so we choose x_2 and multiply by 3. Now there is one $\hat{\phi}_C(x_1)$ which needs to be contracted either with $\hat{\phi}_C(x_3)$ or $\hat{\phi}_C(x_4)$ which are equivalent, so we choose x_3 and multiply by 2. These multiplyings have done away with the 4! in the denominator, and all of the remaining $\hat{\phi}$'s have only one possible contraction. The contractions $\langle p'_A| \hat{\phi}_A(x_1) |0\rangle$ gives $e^{ip'_A\cdot x_1}/\sqrt{2E'_A}$ and likewise for the other three external legs (with $-ip_B \cdot x_1$ for the incoming particles). The contractions of $\hat{\phi}_C(x_1)$ with $\hat{\phi}_C(x_3)$ gives $\langle 0| T\hat{\phi}_C(x_1)\hat{\phi}_C(x_3) |0\rangle = D_F(x_1 - x_3, m_C)$. Thus we have



$$\begin{aligned}\langle \vec{p}'_A\vec{p}'_B| \hat{S}^{(4)} |\vec{p}_A\vec{p}_B\rangle &= (-ig)^4 \int dx_1 \int dx_2 \int dx_3 \int dx_4 e^{i(p'_A-p_B)\cdot x_1} e^{i(p'_B-p_A)\cdot x_3} \\ &\quad D_F(x_1 - x_3, m_C) D_F(x_3 - x_4, m_A) \\ &\quad D_F(x_1 - x_3, m_B) D_F(x_1 - x_3, m_C)\end{aligned}$$

But each of the Feynman propagators can be written in terms of its Fourier transform, $D_F(y) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot y} \tilde{D}_F(q)$, so we have

$$\begin{aligned}\langle \vec{p}'_A\vec{p}'_B| \hat{S}^{(4)} |\vec{p}_A\vec{p}_B\rangle &= (-ig)^4 \int d^4x_1 \int d^4x_2 \int d^4x_3 \int d^4x_4 e^{i(p'_A-p_B)\cdot x_1} e^{i(p'_B-p_A)\cdot x_2} \\ &\quad \int d^4q \int d^4q' \int d^4k \int d^4k' e^{-iq\cdot(x_1-x_3)} e^{-iq'\cdot(x_4-x_2)} e^{-ik\cdot(x_3-x_4)} \\ &\quad e^{ik'\cdot(x_3-x_4)} \tilde{D}_F(q, m_C) \tilde{D}_F(q', m_C) \tilde{D}_F(k, m_A) \tilde{D}_F(k', m_B) \\ &= \int d^4q \int d^4q' \int d^4k \int d^4k' \tilde{D}_F(q, m_C) \tilde{D}_F(q', m_C) \tilde{D}_F(k, m_A) \\ &\quad \tilde{D}_F(k', m_B) \quad \int d^4x_1 \int d^4x_2 \int d^4x_3 \int d^4x_4 \, e^{-ix_1\cdot(q-p'_A+p_B)} \\ &\quad e^{-ix_2\cdot(-q'-p'_B+p_A)} e^{-ix_3\cdot(-q+k-k')} e^{-ix_4\cdot(q'+k'-k)}\end{aligned}$$

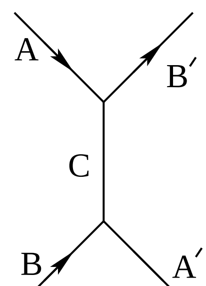
The x integrals each give $(2\pi)^4\delta^4()$ for a linear combination of momenta. The last two give $q = k - k' = q'$, and then the other two give $\delta^4(p'_A + p'_B - p_A - p_B)$, overall momentum conservation, and force $q = p'_A - p_B$ which means $k' = k - q = k - p'_A + p_B$. Thus the q', q and k' integrals are done by imposing the delta functions, but there remains the integral over k,

$$\langle \vec{p}'_A \vec{p}'_B | \, \hat{S}^{(4)} \, | \vec{p}_A \vec{p}_B \rangle = i(2\pi)^4 \delta^4(p'_A + p'_B - p_A - p_B)\mathcal{M},$$

with

$$i\mathcal{M} = (-ig)^4 \tilde{D}_F(q, m_C) \tilde{D}_F(q, m_C) \int \frac{d^4k}{(2\pi)^4} \tilde{D}_F(k, m_A) \tilde{D}_F(k - p'_A + p_B, m_B)$$

The Propagator



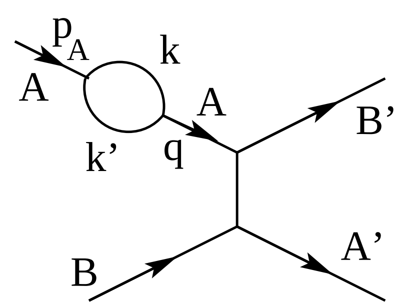
[6.3 (e)]

We see that this diagram can be considered a correction to the simple C exchange tree diagram of Figure 6.3e, in which the simple propagator $\dfrac{i}{q^2 - m_C^2 + i\epsilon}$ has been replaced by $\dfrac{i}{q^2 - m_C^2 + i\epsilon} \left(-i\Pi_C^{[2]}(q^2) \right) \dfrac{i}{q^2 - m_C^2 + i\epsilon}$, where $\Pi_C(q^2)$ is the self energy of the C particle. ($\Pi^{[2]}$ is the contribution to it of second order in g.)

$$\Pi_C^{[2]}(q^2) = -ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_A^2 + i\epsilon} \frac{i}{(q-k)^2 - m_B^2 + i\epsilon}$$

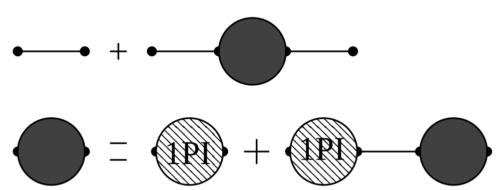
These "self-energy" corrections present a lot of difficult issues in understanding how perturbation theory can possibly work. The idea, of course, is that if g can be considered small, the $\mathcal{O}(g^4)$ term we are calculating is a small correction to the $\mathcal{O}(g^2)$ term we calculated in Lecture 9. But that depends on the integral multiplying g^2 in $\Pi_C^{[2]}(q^2)$. We have learned how to handle the poles at $k^2 = m_A^2$, and at least for the spacelike q that we have for C exchange, after deforming the contours correctly, the integrand is well defined. There are issues for timelike $q^2 > (m_A + m_B)^2$, which affect the corrections to the resonance diagram Fig. 6.3d, but are actually essential for keeping the S matrix a unitary operator. But one immediate difficulty is that the integral *diverges*! If we look at large values of k^μ, or more precisely of $\vec{k}$ and ik^0 after deforming the contour so k^0 is imaginary, we see that the denominator goes like k^4, but if we write the four-dimensional integral over k^μ as a hyperangular integral times $\int_0^\infty k^3 dk$, we see that we have a logarithmically diverging integral $\sim \int dk/k$. So the coefficient of the small g^2 is ∞! This brings up the necessity for the ideas of regularization and renormalization, which we will get to in a bit. But before we do, I want to discuss an issue which would arise even if $\Pi(q^2)$ were finite.

We first saw the diagram with $\Pi(q^2)$ in the propagation of $A \to A'$ in Fig. 6.3c, but we ignored it because it did not contribute to the scattering. But one of the other diagrams which enters at $\mathcal{O}(g^4)$ includes a self-energy bubble on an external leg. The propagator shown with momentum q has a denominator $q^2 - m_A^2$. But the two vertices on the self-energy give $\delta^4(p_A - k - k')$ and $\delta^4(q - k - k')$, which means that $q^2 = p_A^2 = m_A^2$, and the denominator vanishes! This is not just a pole we are integrating around, as q is set at p_A, a fixed external momentum.



This diagram actually gives us a clue about how we should be handling these corrections to the propagator. Recall that our argument for the S matrix is based on the idea that isolated one-particle states of the full interacting Hamiltonian satisfy the free Klein-Gordon equation, basically $p^2 = m^2$, which is the same as the equation of motion one gets from the bare Hamiltonian $\hat{\mathcal{H}}_0$. But is it really the same equation? The single particle has a mass m which might not be the parameter[1] m_0 of the Hamiltonian, because as we see, the rules for the invariant amplitude $\mathcal{M}(p \to p)$ include interactions.

If we ask what the full amplitude is, we would have to add to the bare propagator $i/(p^2 - m_0^2 + i\epsilon)$ all sorts of connected diagrams which we may represent by the dark blob, the sum of all Feynman diagrams between those two vertices. Among these diagrams, there are some called *one-particle irreducible*, or 1PI, which cannot be disconnected by cutting a single line. An arbitrary diagram in the dark blob can be written as some number of 1PI diagrams connected



[1]Until now, we have just called this parameter m, but now we wish to distinguish it, and call it the *bare mass*.

in a line with propagators. If we call the value of the sum of all 1PI diagrams $-i\Pi(q^2)$, we see that the full propagator replaces $\tilde{D}_F(q^2)$ by

$$\begin{aligned}
&\tilde{D}_F(q^2) + \tilde{D}_F(q^2)(-i\Pi(q^2))\tilde{D}_F(q^2) \\
&\qquad + \tilde{D}_F(q^2)(-i\Pi(q^2))\tilde{D}_F(q^2)(-i\Pi(q^2))\tilde{D}_F(q^2) + \ldots \\
&= \tilde{D}_F(q^2) \sum_{n=0}^{\infty} \Big((-i\Pi(q^2))\tilde{D}_F(q^2)\Big)^n = \frac{\tilde{D}_F(q^2)}{1 - \Big(-i\Pi(q^2)\tilde{D}_F(q^2)\Big)} \\
&= \frac{i}{q^2 - m_0^2 - \Pi(q^2)}.
\end{aligned}$$

So now if we ask what the mass of this object is, we would say that its mass is determined by the value of q^2 for which the denominator vanishes, which is to say,

$$m^2 = m_0^2 + \Pi(m^2).$$

We see that the self-energy insertions of the 1PI diagrams gives an $\mathcal{O}(g^2)$ contribution to the *shift* of the mass of the particle.

Now of course when we measure the mass of a particle by observing the relation between its energy and its momentum, we are measuring this mass m of the full propagator, not the bare mass m_0 it would have if there were no interactions, including self-interactions. That means that the $e^{\pm ip\cdot x}$ associated with the external lines already include all these insertions, and we should *amputate* our set of Feynman diagrams to exclude those with self-energy insertions on the external lines.

Next time:

Field Strength Renormalization