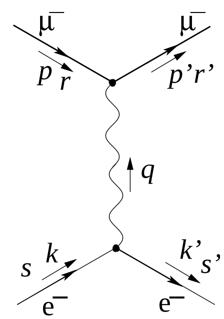
Physics 613 Lecture 13 March 11, 2014

Coulomb Scattering, Form Factors

Unlike Fermion Scattering

Consider the scattering of two different fermions, an electron with charge $-e$ and mass m_e, entering with momentum $\vec{k}$ and another fermion X with charge Q and mass m_X. Particle X must not be an electron or a positron (more on those cases later). In the diagram, it is a μ^-, but for now let it be arbitrary.



The incoming state is $|i\rangle = \sqrt{2E_e}\sqrt{2E_2}c^\dagger_{e\,s}(\vec{k})c^\dagger_{X\,r}(\vec{p})\,|0\rangle$ and the final state is $\langle f| = \sqrt{2E_1'}\sqrt{2E_2'}\,\langle 0|\,c_{1\,s_1'}(\vec{k}_1')c_{2\,s_2'}(\vec{p}')$, and the interaction hamiltonian density is $\mathcal{H}_e + \mathcal{H}_X = \left[-e\bar\psi_e(x)\gamma^\mu\psi_e(x) + Q\bar\psi_X(x)\gamma^\mu\psi_X(x)\right]A_\mu(x)$. The lowest order in the expansion of the exponential that can contribute to scattering needs one copy of $\mathcal{H}_e$ and one of $\mathcal{H}_X$. Thus there is only the one Feynman diagram.

The μ line gives us $-iQ\bar{u}_X(\vec{p}\,',r')\gamma^\mu u_X(\vec{p},r)$, the e^- line gives $ie\bar{u}_e(\vec{k}\,',s')\gamma^\nu u_e(\vec{k},s)$, the photon line gives $-i(g_{\mu\nu} - (1-\xi)q_\mu q_\nu/q^2)/(q^2+i\epsilon)$ so all together

$$i\mathcal{M} = -ieQ\frac{\bar{u}_X(\vec{p}\,',r')\gamma^\mu u_X(\vec{p},r)\bar{u}_e(\vec{k}\,',s')\gamma^\nu u_e(\vec{k},s)(g_{\mu\nu} - (1-\xi)q_\mu q_\nu/q^2)}{q^2+i\epsilon} \tag{1}$$

where $q^\mu = (p'-p)^\mu$. The first thing to comment on is the term proportional to $1-\xi$, which involves a factor of $-iq\bar{u}(\vec{p}\,',r')\gamma^\mu(p_\mu - p'_\mu)u(\vec{p},r)$. But $\gamma^\mu p_\mu u(\vec{p},r) = mu(\vec{p},r)$ as u satifies the Dirac equation $(\not p - m)u(p) = 0$, and also $\bar{u}(\vec{p}\,',r')\gamma^\mu p'_\mu = m\bar{u}(\vec{p}\,',r')$. Thus the gauge dependent term vanishes. This leaves

$$i\mathcal{M} = -ieQ\frac{\bar{u}_X(\vec{p}\,',r')\gamma^\mu u_X(\vec{p},r)\bar{u}_e(\vec{k}\,',s')\gamma_\mu u_e(\vec{k},s)}{q^2}.$$

This vanishing is a general feature guaranteed by the Ward identity, though in more complicated diagrams it may be only in a sum over diagrams that the gauge-dependent term vanishes.

If we calculate the cross section in the center of mass (see Lecture 9 page 6),

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm cm} = \frac{1}{64\pi^2 s}\left|\mathcal{M}\right|^2$$

but calculating

$$\left|\mathcal{M}\right|^2 = \frac{e^2Q^2}{q^4}\left|\bar{u}_X(\vec{p}\,',r')\gamma^\mu u_X(\vec{p},r)\bar{u}_e(\vec{k}\,',s')\gamma_\mu u_e(\vec{k},s)\right|^2$$

is not easy to do in general.

e^- Coulomb scattering

Let us consider the limit where the X particle mass is much larger than the electron's or any of the momenta. In either the lab or center-of-mass frame (which are the same in the limit $M \to \infty$), $p^\mu/M = p'^\mu/M \sim (1,0,0,0)$. In evaluating $\mathcal{M}$ the factor $\bar{u}_X(\vec{p}\,',r')\gamma^\mu u_X(\vec{p},r)$ simplifies greatly. The bottom components of u go to zero, $u \sim \sqrt{2M}\begin{pmatrix}\phi^r\\ 0\end{pmatrix}$, and similarly $\bar{u} \sim \sqrt{2M}(\phi^{r'\,\dagger},0)$. Then $\bar{u}_X(\vec{p}\,',r')\gamma^\mu u_X(\vec{p},r) = 2M\delta^\mu_0\delta_{rr'}$. In the center of mass frame $E_e = E_e'$ and $|\vec{k}| = |\vec{k}\,'| = E_e v$ and then

$$\begin{aligned}
\bar{u}_e(\vec{k}\,',s')\gamma_0 u_e(\vec{k},s) &= u_e^\dagger(\vec{k}\,',s')u_e(\vec{k},s) = (E_e+m_e)\phi^{s'\,\dagger}\left(1+\frac{(\vec\sigma\cdot\vec{k}\,')(\vec\sigma\cdot\vec{k})}{(E_e+m_e)^2}\right)\phi^s \\
&= \phi^{s'\,\dagger}\frac{(E_e+m_e)^2 + \vec{k}\,'\cdot\vec{k} + i\vec\sigma\cdot(\vec{k}\,'\times\vec{k})}{E_e+m_e}\phi^s \\
&= \phi^{s'\,\dagger}\left[E_e+m_e) + (E_e-m_e)\Big(\hat{v}'\cdot\hat{v} + i\vec\sigma\cdot(\hat{v}'\times\hat{v})\Big)\right]\phi^s
\end{aligned}$$

where here $\hat{v}$ is a unit vector in the $\vec{k}$ direction, not an operator. So we have the invariant amplitude

$$\mathcal{M} = -\frac{2MeQ}{q^2}\delta_{rr'}\phi^{s'\,\dagger}\left[E_e+m_e) + (E_e-m_e)\Big(\hat{v}'\cdot\hat{v} + i\vec\sigma\cdot(\hat{v}'\times\hat{v})\Big)\right]\phi^s.$$

If we do not detect the final spins, we have

$$\begin{aligned}\sum_{r',s'}\frac{d\sigma}{d\Omega}\bigg|_{\rm cm} &= \frac{1}{64\pi^2 s}\left(\frac{2MeQ}{q^2}\right)^2\phi^{s\,\dagger}\left[E_e+m_e+(E_e-m_e)\Big(\hat v'\cdot\hat v - i\vec\sigma\cdot(\hat v'\times\hat v)\Big)\right]\\ &\qquad\times\sum_{s'}\phi^{s'}\phi^{s'\,\dagger}\left[E_e+m_e+(E_e-m_e)\Big(\hat v'\cdot\hat v + i\vec\sigma\cdot(\hat v'\times\hat v)\Big)\right]\phi^s\\ &= \frac{e^2Q^2}{16\pi^2}\frac{1}{q^4}\left[(E_e+m_e+(E_e-m_e)\cos\theta)^2+\Big(E_e-m_e)\sin\theta\Big)^2\right]\\ &= \frac{e^2Q^2}{64\pi^2}\frac{E^2}{k^4}\frac{1-v^2\sin^2(\theta/2)}{\sin^4(\theta/2)}\end{aligned}$$

This is the Mott cross section. Usually our "infinitely heavy" particle could be a nucleus of charge Ze, in which case we have

$$\sum_{r',s'}\frac{d\sigma}{d\Omega} = (Z\alpha)^2\frac{E^2}{4k^4}\frac{1-v^2\sin^2(\theta/2)}{\sin^4(\theta/2)}$$

Back to Finite Mass Particles

Let us return to the difficult $|\mathcal{M}|^2$ calculation in general. Actually the experiment is not easy to do either, because it means colliding two polarized beams, or one polarized beam into a polarized target, and measuring the polarizations of the outgoing particles. It is much easier to use unpolarized beams, each half spin up and half spin down, so we should sum over initial spins and divide by 4. Also it is easier to use detectors which don't care about spin, so detect both, and we should sum over spins of each of the final particles.

Surprisingly, that actually makes things easier for the phenomenologist as well. We have[1]

$$\begin{aligned}\frac14\sum_{r,r',s,s'}\mathcal{M}\mathcal{M}^* &= \frac{e^2Q^2}{q^4}M^{\mu\rho}L_{\mu\rho}\\ \text{with}\quad M^{\mu\rho} &= \frac12\sum_{r,r'}\bar u_X(\vec p\,',r')\gamma^\mu u_X(\vec p,r)(\bar u_X(\vec p\,',r')\gamma^\rho u_X(\vec p,r))^*\\ \text{and}\qquad L_{\mu\rho} &= \frac12\sum_{s,s'}\bar u_e(\vec k\,',s')\gamma_\mu u_e(\vec k,s)(\bar u_e(\vec k\,',s')\gamma_\rho u_e(\vec k,s))^*.\end{aligned}$$

[1]Please note how the sum over μ is $\mathcal{M}$ is handled for the square!

Note that $(\bar u(\vec p\,',r')\gamma^\rho u(\vec p,r))^* = \bar u(\vec p,r)\gamma^\rho u(\vec p\,',r')$, where we have used $(\gamma^0\gamma^\rho)^\dagger = \gamma^0\gamma^\rho$. Thus

$$M^{\mu\rho} = \frac12\sum_{r'}\bar u_X(\vec p\,',r')\gamma^\mu\left(\sum_r u_X(\vec p,r)(\bar u_X(\vec p,r))\right)\gamma^\rho u_X(\vec p\,',r').$$

But as we saw in homework 6 problem 2, the term in () is $\not p + M$, and writing out $M^{\mu\rho}$ with some spinor indices showing, is now

$$\begin{aligned}M^{\mu\rho} &= \frac12\sum_{r'}(\bar u_X(\vec p\,',r'))_a\left(\gamma^\mu(\not p+M)\gamma^\rho\right)_{ab}(u_X(\vec p\,',r'))_b\\ &= \frac12\sum_{r'}(u_X(\vec p\,',r'))_b(\bar u_X(\vec p\,',r'))_a\left(\gamma^\mu(\not p+M)\gamma^\rho\right)_{ab}(u_X(\vec p\,',r'))_b\\ &= \frac12\,\mathrm{Tr}[(\not p'+M)\gamma^\mu(\not p+M)\gamma^\rho].\end{aligned}$$

Of course the same calculation gives

$$L_{\mu\rho} = \frac12\,\mathrm{Tr}[(\not k'+m)\gamma_\mu(\not k+m)\gamma_\rho].$$

Now it is time for "trace technology" to learn how to evaluate traces of products of gamma matrices. From $\{\gamma_5,\gamma^\mu\}=0$ we have $\gamma_5\prod_{j=1}^n\gamma^{\mu_j} = (-1)^n\left(\prod_{j=1}^n\gamma^{\mu_j}\right)\gamma_5$. Premultiplying by γ_5^{-1} and taking the trace says

$$\begin{aligned}\mathrm{Tr}\prod_{j=1}^n\gamma^{\mu_j} &= (-1)^n\,\mathrm{Tr}\left[\gamma_5^{-1}\left(\prod_{j=1}^n\gamma^{\mu_j}\right)\gamma_5\right] = (-1)^n\,\mathrm{Tr}\left[\left(\prod_{j=1}^n\gamma^{\mu_j}\right)\gamma_5\gamma_5^{-1}\right]\\ &= (-1)^n\,\mathrm{Tr}\left[\left(\prod_{j=1}^n\gamma^{\mu_j}\right)\right],\end{aligned}$$

where the second equality is due to the cyclicity of the trace. We see that the trace of an odd number of γ^{μ_j}'s vanishes. From tracing $\gamma^\mu\gamma^\nu = 2g^{\mu\nu}\mathbb{I}-\gamma^\nu\gamma^\mu$ we see that $\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = 8g^{\mu\nu}-\mathrm{Tr}\,\gamma^\nu\gamma^\mu = 8g^{\mu\nu}-\mathrm{Tr}\,(\gamma^\mu\gamma^\nu)$ so

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}.$$

Similarly

$$\begin{aligned}\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\sigma\right] &= \mathrm{Tr}\left[\left(2g^{\alpha\beta}-\gamma^\beta\gamma^\alpha\right)\gamma^\rho\gamma^\sigma\right]\\ &= 2g^{\alpha\beta}\,\mathrm{Tr}\left[\gamma^\rho\gamma^\sigma\right]-\mathrm{Tr}\left[\gamma^\beta\left(2g^{\alpha\rho}-\gamma^\rho\gamma^\alpha\right)\gamma^\sigma\right]\\ &= 2g^{\alpha\beta}\,\mathrm{Tr}\left[\gamma^\rho\gamma^\sigma\right]-2g^{\alpha\rho}\,\mathrm{Tr}\left[\gamma^\beta\gamma^\sigma\right]+\mathrm{Tr}\left[\gamma^\beta\gamma^\rho\left(2g^{\alpha\sigma}-\gamma^\sigma\gamma^\alpha\right)\right]\\ &= 8g^{\alpha\beta}g^{\rho\sigma}-8g^{\alpha\rho}g^{\beta\sigma}+8g^{\alpha\sigma}g^{\beta\rho}-\mathrm{Tr}\left[\gamma^\beta\gamma^\rho\gamma^\sigma\gamma^\alpha\right]\end{aligned}$$

But the last trace is the same as the left hand side, so

$$\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\sigma\right] = 4g^{\alpha\beta}g^{\rho\sigma} - 4g^{\alpha\rho}g^{\beta\sigma} + 4g^{\alpha\sigma}g^{\beta\rho} \qquad (2)$$

Whew again. The same trick can be used to evaluate the trace of a product of $2n$ γ^{α_j}'s, but there will be $(2n-1)!!$ terms, each a product of n $g^{\alpha_j\alpha_k}$'s. Fortunately, we won't need more than 4, unless we decide to evaluate spin-resolved cross sections.

Now we can write

$$\begin{aligned} M^{\mu\rho} = \frac{1}{2}\mathrm{Tr}[(\not p' + M)\gamma^\mu(\not p + M)\gamma^\rho] &= 2p'^\mu p^\nu + 2p'^\nu p^\mu - 2p'\cdot p g^{\mu\nu} + 2M^2 g^{\mu\nu} \\ &= 2p'^\mu p^\nu + 2p'^\nu p^\mu + q^2 g^{\mu\nu}, \end{aligned}$$

where in the last expression we used $q^2 = (p'-p)^2 = (p')^2 - 2p\cdot p' + p^2 = 2(m^2 - p\cdot p'$. and this gives

$$\begin{aligned} \frac{1}{4}\sum_{r,r',s,s'}\mathcal{M}\mathcal{M}^* &= \frac{e^2Q^2}{q^4}\left[2p'^\mu p^\nu + 2p'^\nu p^\mu + q^2 g^{\mu\nu}\right] \\ &\quad \times\left[2k'_\mu k_\nu + 2k'_\nu k^\mu + q^2 g_{\mu\nu}\right] \\ &= \frac{4e^2Q^2}{q^4}\left[2p'\cdot k' p\cdot k + 2p'\cdot k p\cdot k' + (m^2+M^2)q^2\right] \end{aligned}$$

Proton Form Factors

In the process just described, the particle X could have been the muon, but it could also be the proton or even a ${}^7_3$Li nucleus, except that we have assumed X is a point Dirac fundamental particle, which these are not. Nontheless, a proton or a ${}^7_3$Li nucleus in its ground state is a fermion in a state completely described by its momentum and spin, and so we can do the calculation of scattering an electron (or muon) off it in the same way, with one exception. In the interaction hamiltonian $\mathcal{H}_X = \int d^3x A_\mu(x) J^\mu(x)$, we cannot assume the charge density and current $J^\mu(x)$ is given by the Dirac form for an elementary particle. The factor $\bar u_X(\vec p\,', r')\gamma^\mu u_X(\vec p, r)$ in (1) needs to be replaced by $\bar u_X(\vec p\,', r')J^\mu u_X(\vec p, r)$. What can we say about $J^\mu(x)$? By translation invariance, $J^\mu(x^\rho) = e^{ix^\rho\hat P_\rho}J_\mu(0)e^{-ix^\rho\hat P_\rho}$ so the matrix element $e^{i(p'-p)_\rho x^\rho}J_\mu(0)$ must transform as a lorentz vector which could be a function

of $q = p' - p$. The simplest possibility is $e\gamma^\mu\mathcal{F}_1(q^2)$, but we might also consider $C(q^2)q^\mu$ or[2] $ie\kappa\mathcal{F}_2(q^2)\sigma^{\mu\nu}q_\nu/2M$. We are discarding as possibilities $\gamma_5 q^\mu$ or $\gamma^\mu\gamma^5$ because we assume parity conservation of the electromagnetic interaction. But we also need current conservation, $\partial_\mu J^\mu(x) = 0$ or $q_\mu\tilde J^\mu(q) = 0$. Note $q_\mu\bar u_X(\vec p\,', r')\left[e\gamma^\mu\mathcal{F}_1(q^2) + ie\kappa\mathcal{F}_2(q^2)\sigma^{\mu\nu}q^\nu/2M + C(q^2)q^\mu\right]u_X(\vec p, r) = q^2C(q^2)2M\delta_{rr'}$, because $q_\mu\gamma^\mu = \not p' - \not p \sim M - M = 0$ and $q_\mu\sigma^{\mu\nu}q_\nu = 0$. Thus the $C(q^2)$ vanishes, and the correct factor is

$$e\bar u_P(\vec p\,', r')\left[\mathcal{F}_1(q^2)\gamma^\mu + i\kappa\mathcal{F}_2(q^2)\sigma^{\mu\nu}q_\nu/2M\right]u_P(\vec p, r).$$

$\mathcal{F}_1(q^2)$ and $\mathcal{F}_2(q^2)$ are known as the *form factors* for the proton.

Note that for a proton at rest, the $q=0$ value of the time component of this is the integral of the charge density over all space, and so must be the total charge e of the proton, and we must have $\mathcal{F}_1(0) = 1$. If we consider an external constant magnetic field $\vec B = \vec\nabla\times\vec A$, with $\vec A = \frac{1}{2}\vec B\times\vec x$,

$$\begin{aligned} H_I &= -\int d^3x\vec A(x)\cdot\vec J(x) = -\frac{1}{2}\int d^3x\int\frac{d^3q}{(2\pi)^3}\left[\vec B\times\left(-i\vec\nabla_q e^{i\vec q\cdot\vec x}\right)\right]\cdot\tilde{\vec J}(\vec q) \\ &= -i\frac{1}{2}\int d^3x\int\frac{d^3q}{(2\pi)^3}e^{i\vec q\cdot\vec x}\vec B\times\left[\vec\nabla_q\cdot\tilde{\vec J}(\vec q)\right] = i\frac{1}{2}\,\vec B\cdot\nabla_q\times\tilde{\vec J}(\vec q)\Big|_{q=0}. \end{aligned}$$

Thus we need the term linear in q in $\tilde J_j$ for a proton at rest. It helps to first derive the *Gordon identity*

$$\bar u(\vec p\,')\gamma^\mu u(\vec p) = \bar u(\vec p\,')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(\vec p).$$

To see this, note

$$\begin{aligned} \bar u(p')\sigma^{\mu\nu}q_\nu u(p) &= \frac{i}{2}\bar u(p')\left[\gamma^\mu(\not p' - \not p) - (\not p' - \not p)\gamma^\mu\right]u(p) \\ &= \frac{i}{2}\bar u(p')\left[\{\gamma^\mu, \not p'\} - 2\not p'\gamma^\mu - 2\gamma^\mu\not p + \{\not p, \gamma^\mu\}\right]u(p) \\ &= i\bar u(p')\left[p'^\mu - \not p'\gamma^\mu - \gamma^\mu\not p + p^\mu\right]u(p) \\ &= i\bar u(p')\left[p'^\mu + p^\mu - 2m\gamma^\mu\right]u(p). \end{aligned}$$

[2] In 615 or in Peskin and Schroeder, the second form factor $F_2(q^2)$ includes the constant κ.

As $\bar{u}(q,r')u(0,r) = 2M\delta_{rr'} + \mathcal{O}(q^2)$, the term linear in $\vec{q}$ from $\mathcal{F}_1$ adds to $\kappa\mathcal{F}_2$, and we have

$$H_I = \frac{i}{2}\epsilon_{jk\ell}B_j\bar{u}(\vec{0},r')\frac{i\sigma^{\ell}{}_k}{2M}u(\vec{0},r)\left(\mathcal{F}_1{+}\kappa\mathcal{F}_2\right) = -\frac{e\left(\mathcal{F}_1{+}\kappa\mathcal{F}_2\right)}{2M}\,\vec{B}\cdot\bar{u}(\vec{0},r')\vec{\sigma}u(\vec{0},r)$$

But the classical expression is $H = -\vec{\mu}\cdot\vec{B}$, so the magnetic moment is

$$\vec{\mu} = \frac{e}{M}\left[\mathcal{F}_1(0) + \kappa\mathcal{F}_2(0)\right]\vec{S},$$

where $\vec{S} = \frac{1}{2}\bar{u}(\vec{0},r')\vec{\sigma}u(\vec{0},r)$ is the spin of the proton. The generic description of magnetic moment is $\vec{\mu} = g\,\dfrac{e}{2M}\vec{S}$, so we have

$$g = 2\left(\mathcal{F}_1(0) + \kappa\mathcal{F}_2(0)\right).$$

Our calculation for an elementary fermion assumed $\mathcal{F}_1(q^2) \equiv 1$ and $\kappa\mathcal{F}_2(q^2) \equiv 0$ for all q^2. Note that this gives $g = 2$, which is twice what would be expected for a classical non-relativistic rigidly rotating charge distribution. For a composite object, we still have $\mathcal{F}_1(0) = 1$ because we have already factored out the total charge Q, but we might expect a variation with q^2 for an extended charge distribution. For $\kappa\mathcal{F}_2(0)$ there is no restriction, and indeed the proton has $\kappa\mathcal{F}_2(0) = 1.79$ with a total $g/2 = 2.79$, quite far from the elementary particle value. But it should be noted that the Dirac value $g = 2$ is not exact even for the elementary electron, because, as we shall see, higher order Feynman diagrams give a small correction, beginning with $\Delta g = \alpha/4\pi$, but which is now known to better than 13 significant digits, the result of decades of heroic work by both experimentalists and theorists.