Physics 613 Lecture 12 March 6, 2014

More C, Time Reversal, Feynman Rules

More on C

From $\hat{C}\hat{\psi}(\vec{x},t)\hat{C}^{-1} = i\gamma^2(\hat{\psi}^\dagger)^T(\vec{x},t)$ hermitean conjugation gives $\hat{C}\hat{\psi}^\dagger(\vec{x},t)\hat{C}^{-1} = (\hat{\psi})^T(\vec{x},t)i\gamma^2$. The transposes can be a bit confusing when we deal with bilinears, so let's rewrite these with spinor indices showing (and implicit sums over ones repeated in a term). [Nothing is going to happen to the $\vec{x}$ and t, so we will leave these implicit.]

$$\hat{C}\hat{\psi}_a\hat{C}^{-1} = i\gamma^2_{ab}(\hat{\psi}^\dagger)_b, \qquad \hat{C}\hat{\psi}^\dagger_a\hat{C}^{-1} = (\hat{\psi})_b i\gamma^2_{ba}.$$

Now consider $\hat{\bar{\psi}}\Gamma\hat{\psi} = \hat{\psi}^\dagger_a(\gamma^0\Gamma)_{ab}\hat{\psi}_b$ for an arbitrary spinor matrix Γ. We have

$$\begin{aligned}
\hat{C}\hat{\psi}^\dagger_a(\gamma^0\Gamma)_{ab}\hat{\psi}_b C^{-1} &= \hat{\psi}_c i\gamma^2_{ca}(\gamma^0\Gamma)_{ab} i\gamma^2_{bd}(\hat{\psi}^\dagger)_d \\
&= -i^2\hat{\psi}^\dagger_d\gamma^2_{bd}(\gamma^0\Gamma)_{ab}\gamma^2_{ca}\hat{\psi}_c + i^2\gamma^2_{ca}\gamma^2_{bd}(\gamma^0\Gamma)_{ab}\delta_{cd}\delta^3(0) \\
&= \hat{\psi}^\dagger\gamma^{2\,T}\Gamma^T\gamma^{0\,T}\gamma^{2\,T}\hat{\psi} - \mathrm{Tr}(\gamma^2\gamma^0\Gamma\gamma^2)\delta^3(0) \\
&= \hat{\bar{\psi}}\gamma^0\gamma^2\Gamma^T\gamma^0\gamma^2\hat{\psi} + \mathrm{Tr}(\gamma^0\Gamma)\delta^3(0)
\end{aligned}$$

The trace term only affects $\Gamma \propto \gamma^0$, and we should point out here that the bilinear $\hat{\psi}^\dagger(\vec{x},t)\hat{\psi}(\vec{x},t)$ is inherently ill-defined because the anticommutator gives the $\delta^3(0)$. We will define the *normal ordered product* of fields and creation and annihilation operators, $\mathbf{N}(\prod_j \Phi_j)$ to be what you get by writing out $\prod_j \Phi_j$ in terms of creation and annihilation operators, moving all the annihilation operators to the right of all the annihilation operators, treating them as commuting or anticommuting as appropriate, dropping all the terms you might get from the commutators or anticommutators. Note that this is what we did to drop the infinite terms in $\hat{H}$ and $\hat{Q}$ in Lecture 10. As C does not interchange the order of creation with annihilation operators, if the original bilinear was normal ordered, the final one should be as well. So we drop the infinite c-number term and treat all these bilinears as normal ordered.

Thus we have

$$\hat{C}\hat{\bar{\psi}}\Gamma\hat{\psi}\hat{C}^{-1} = \hat{\bar{\psi}}\gamma^0\gamma^2\Gamma^T\gamma^0\gamma^2\hat{\psi}.$$

As $(\gamma^0\gamma^2)^2 = 1$, we see that charge conjugation will act according to how Γ transposes and commutes with $\gamma^0\gamma^2$. Any spinor matrix is a linear combination of $\mathbb{1}$, γ^μ, $[\gamma^\mu,\gamma^\nu]$, $\gamma^\mu\gamma_5$, and γ_5. The γ^μ are symmetric for $\mu = 0$ and $\mu = 2$, and antisymmetric otherwise, and γ_5 is symmetric. γ^0 and γ^2 anticommute with $\gamma^0\gamma^2$ and the others, including γ_5, commute. The product on two of these, γ^μ's or γ_5, picks up an extra -1 under transposition as they all anticommute. So the upshot is

$$\hat{C}\hat{\bar{\psi}}\hat{\psi}\hat{C}^{-1} = \hat{\bar{\psi}}\hat{\psi}, \quad \hat{C}\hat{\bar{\psi}}\gamma^\mu\hat{\psi}\hat{C}^{-1} = -\hat{\bar{\psi}}\gamma^\mu\hat{\psi}, \quad \hat{C}\hat{\bar{\psi}}[\gamma^\mu,\gamma^\nu]\hat{\psi}\hat{C}^{-1} = -\hat{\bar{\psi}}[\gamma^\mu,\gamma^\nu]\hat{\psi}$$

$$\hat{C}\hat{\bar{\psi}}\gamma^\mu\gamma_5\hat{\psi}\hat{C}^{-1} = \hat{\bar{\psi}}\gamma^\mu\gamma_5\hat{\psi}, \qquad \hat{C}\hat{\bar{\psi}}\gamma_5\hat{\psi}\hat{C}^{-1} = \hat{\bar{\psi}}\gamma_5\hat{\psi}.$$

Time Reversal

We expect that a symmetry should commute with the Hamiltonian, and we have seen that it does commute with the unitary operators $\hat{P}$ and $\hat{C}$. But when it comes to time reversal

$$\frac{\partial\hat{\phi}(\vec{x},t)}{\partial t} = i[\hat{H},\hat{\phi}(\vec{x},t)] \tag{1}$$

gives us trouble. As we expect

$$\hat{\phi}_T(\vec{x},t) = \hat{T}\hat{\phi}(\vec{x},t)\hat{T}^{-1} = \hat{\phi}(\vec{x},-t), \tag{2}$$

applying $\hat{T}$ to (1) would seem to give

$$\frac{\partial\hat{\phi}(\vec{x},-t)}{\partial t} = i[\hat{T}\hat{H}\hat{T}^{-1},\hat{\phi}(\vec{x},-t)] = -i[\hat{H},\hat{\phi}(\vec{x},-t)]. \tag{3}$$

But that would mean $\hat{T}$ does not commute with $\hat{H}$, in fact it changes its sign, which would give a Hamiltonian unbounded from below. So we cannot have $\hat{T}$ be a symmetry represented by a unitary operator.

But there is a way out. The i in Schrödinger's equation was an arbitrary choice, and we would have the same physics if Martians choose the opposite sign and complex-conjugated every state and every operator. So consider that $\hat{T}$ is an *antiunitary* operator, $\hat{T} = \hat{U}_T\mathbf{K}$, where $\hat{U}_T$ is a unitary operator and $\mathbf{K}$ complex-conjugates everything to its right. Then the i in the second expression in (3) is replaced by $TiT^{-1} = -i$ and everything is okay.

Expanding the scalar field as usual, we see that

$$\hat{T}\hat{\phi}(\vec{x},t)\hat{T}^{-1}=\int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}}\left[\hat{T}\hat{a}(\vec{k})\hat{T}^{-1}e^{i\omega t-i\vec{k}\cdot\vec{x}}+\hat{T}\hat{b}^\dagger(\vec{k})\hat{T}^{-1}e^{-i\omega t+i\vec{k}\cdot\vec{x}}\right]=\hat{\phi}(\vec{x},-t)$$

provided $\hat{T}\hat{a}(\vec{k})\hat{T}^{-1}=\hat{a}(-\vec{k})$, $\hat{T}\hat{b}^\dagger(\vec{k})\hat{T}^{-1}=\hat{b}^\dagger(-\vec{k})$, which is what we would expect, as the three-momentum should change sign under time-reversal.

For the Dirac field,

$$\begin{aligned}\hat{T}\hat{\psi}(\vec{x},t)\hat{T}^{-1} &= \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}}\sum_s\Big[\hat{T}\hat{c}_s(\vec{k})\hat{T}^{-1}u^*(\vec{k},s)e^{i\omega t-i\vec{k}\cdot\vec{x}} \\ &\qquad +\hat{T}\hat{d}_s^\dagger(\vec{k})\hat{T}^{-1}v^*(\vec{k},s)e^{-i\omega t+i\vec{k}\cdot\vec{x}}\Big] \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}}\sum_s\Big[\hat{T}\hat{c}_s(-\vec{k})\hat{T}^{-1}u^*(-\vec{k},s)e^{-i\omega(-t)+i\vec{k}\cdot\vec{x}} \\ &\qquad +\hat{T}\hat{d}_s^\dagger(-\vec{k})\hat{T}^{-1}v^*(-\vec{k},s)e^{i\omega(-t)-i\vec{k}\cdot\vec{x}}\Big] \\ &= -i\gamma^1\gamma^3\hat{\psi}(\vec{x},-t)\end{aligned}$$

provided

$$\hat{T}\hat{c}_s(\vec{k})\hat{T}^{-1}=\hat{c}_{s'}(-\vec{k}),\qquad \hat{T}\hat{d}_s^\dagger(\vec{k})\hat{T}^{-1}=\hat{d}_{s'}^\dagger(-\vec{k}),$$

where we have used $u^*(-\vec{k},s)=-i\gamma^1\gamma^3u(\vec{k},s')$ $v^*(-\vec{k},s)=-i\gamma^1\gamma^3v(\vec{k},s')$, where, because $-i\gamma^1\gamma^3=\begin{pmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix}$, we see that the spin is flipped, as we should expect because time has been reversed.

When it comes to the bilinears, the behavior under time reversal will depend on how $\gamma^0\Gamma$ commutes with $\gamma^1\gamma^3$, which will give (-1) for $\Gamma=\gamma^1$, γ^3, $[\gamma^\mu,\gamma^\nu]$ for $\mu\nu=01$, 03, 12, and 23, and $\gamma^\mu\gamma_5$ for $\mu=1$, 3. The complex conjugation gives a (-1) for each γ^2. Thus far we have (-1) for $\vec{\gamma}$, $\vec{\gamma}\gamma_5$ and $[\gamma^0,\vec{\gamma}]$. But we also get a (-1) for each explicit i, so it is time to limit ourselves to *hermitean* bilinears. These are $\hat{\bar{\psi}}\Gamma\hat{\psi}$ with $\Gamma=\mathbb{1},i\gamma_5,\gamma^\mu,\gamma^\mu\gamma^5$, and $\sigma^{\mu\nu}:=\frac{i}{2}[\gamma^\mu,\gamma^\nu]$, which are called S, P, V, A, and T, for scalar, pseudoscalar, (polar) vector, axial vector, and tensor respectively. We see that the pseudoscalar, vector, axial vector and tensor change sign, but not the scalar, except that each time component picks up an additional (-1).

Lagrangians are made from bilinears, but we also have ∂_μ, which gets a minus sign for parity if $\mu\neq 0$ and for time reversal for $\mu=0$. To summarize, I copy a table from Peskin and Schroeder,

	$\hat{\bar{\psi}}\hat{\psi}$	$i\hat{\bar{\psi}}\gamma_5\hat{\psi}$	$\hat{\bar{\psi}}\gamma^\mu\hat{\psi}$	$\hat{\bar{\psi}}\gamma^\mu\gamma_5\hat{\psi}$	$\hat{\bar{\psi}}\sigma^{\mu\nu}\hat{\psi}$	∂_μ
$\hat{P}$	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
$\hat{T}$	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
$\hat{C}$	$+1$	$+1$	-1	$+1$	-1	$+1$
$\widehat{CPT}$	$+1$	$+1$	-1	-1	$+1$	-1

where $(-1)^\mu$ means -1 for $\mu\neq 0$, $(-1)^0=1$. We see that under CPT, each Lorentz index picks up a minus sign, and otherwise all bilinears are invariant. But a lagrangian density must be a scalar, and so must have all terms with an even number of Lorentz indices, so all lagrangians we can build this way are PCT invariant.

Real Scattering

In Lectures 7 and 8 we learned quite generally[1] how to calculate scattering cross sections in perturbation theory. We need to make an initial state $|i\rangle$ from creation operators describing the particles in the incoming beams (or static target), and a final state $|f\rangle$ corresponding to the particles passing through our detectors, so that $\langle f|$ will be of the form $\langle 0|$ followed by a product of annihilation operators. Each should be suitably normalized, which requires a $\sqrt{2E}$ for each particle. Between this $\langle f|$ and $|i\rangle$ we place the S matrix

$$\hat{S}=\mathcal{T}\exp\left\{-i\int d^4x\,\hat{\mathcal{H}}_I(x)\right\}.$$

We then expand the exponential to whatever order in perturbation theory we feel up to calculating, which leaves us with a vacuum expectation value

$$\langle 0|\prod(\text{annihilation operators})\prod(\text{quantum fields})\prod(\text{creation operators})\,|0\rangle\,.$$

We break up each field into its creation piece and its annihilation piece, and then commute or anticommute each piece, creations to the left and annihilations to the right, until each one vanishes acting on the $\langle 0|$ or $|0\rangle$. From the (anti)commutators, which we will call contractions, we get c-number functions. For two fields this gives a Feynman propagator, which is the vacuum expectation value of the time-ordered product of the fields, while a field contracted with a creation or annihilation operator gives a fourier-transforming

[1]There are caveats and modifications to come when we get to loops in chapter 10.

exponential and some functions like $u(\vec{k}, s)$, while contracting external creation and annihilation operators gives a momentum delta function which usually gives an irrelevant piece because that particle has not scattered.

This process leaves us with no fields, but only a sum over terms, each of which is a product of contractions and integrals that come from the $\hat{\mathcal{H}}_I(x)$'s, which can be represented by a Feynman diagram. The Feynman diagram has a *vertex* for each $\hat{\mathcal{H}}_I(x)$, a line for each incoming or outgoing particle connected to one vertex, and lines for each contraction of two fields connecting the two vertices involved. For fields which are not self-conjugate, the propagator, whether $D_F(x_1 - x_2) := \langle 0| T\hat{\phi}(x_1)\hat{\phi}^\dagger(x_2) |0\rangle$ or $S_F := \langle 0| T\hat{\psi}(x_1)\hat{\bar{\psi}}(x_2) |0\rangle$, has a direction associated with it, carrying a particle created at x_2 to x_1, or an antiparticle in the opposite direction. So we place an arrow in the direction from x_2 to x_1. Because complex fields can only contract with their conjugate, these directed lines never terminate inside the graph, though they do enter or leave with external particles. Note that this *charge arrow* may not be in the direction forward in time, and in particular an incoming antiparticle has a direction facing away from the interaction, backwards in time. This can be confusing, because we will also be associating a momentum flow with each line, which might be in the opposite direction.

Unless we are considering scattering off a fixed external potential, which the book does in sections 8.1-2, our hamiltonian has no explicit x^μ dependence, and the dependence on the x_j^μ of the integrand is a dependence on differences $x_j^\mu - x_k^\mu$ plus a factor of $\exp[(\sum_f p_f - \sum_i p_i)_\mu x_1^\mu]$, so the integral over x_1 gives an overall $(2\pi)^4\delta^4(\sum_f p_f - \sum_i p_i)$ which we drop from S to define $i\mathcal{M}$, the invariant scattering amplitude. We can expand the propagators in terms of their Fourier transforms $\tilde{D}_F(q_\mu)$ or $\tilde{S}_F(q_\mu)$, which gives us an integration over the momentum of each interior line in the diagram, together with exponentials $e^{\pm iq_\mu x_j^\mu}$, so the integral over the x_j gives a delta function in the sum of incoming momenta (minus outgoing momenta) for the vertex.

The result of all this is a set of Feynman rules in momentum space. If we are discussing Dirac particles with minimal electromagnetic interactions, these are the rules, though the generalization to other $\mathcal{H}_I$ is straightforward.

- Define a momentum for each line, with a momentum-flow arrow, with external lines having their real momentum and momentum conservation at each vertex.

- For each complex field line, start with the outgoing *charge-arrow* and tracing it back to the incoming particle, or, in the case of a loop, at the beginning of one propagator. For a Dirac field,
  - place a $\bar{u}(p_j, s_j)$ if an outgoing particle line, a $\bar{v}(k_j, s_j)$ for an incoming antiparticle, or a "$-$ Tr(" if in the middle of a loop.
  - Then at each vertex with a photon going backwards, place a $-iq\gamma^\mu$.
  - For each internal line traversed insert $\tilde{S}_F(q) = i/(q - m + i\epsilon)$, if the momentum arrow is in the same direction as the charge arrow, or $\tilde{S}_F(-q)$ otherwise.
  - Then when you reach the incoming line, place a $u(k_j, s_j)$ if an incoming particle or a $v(p_j, s_j)$ if an outgoing antiparticle. Or, if the path is a loop, when you get back to the starting point, close the trace with a). Don't forget the last propagator.
- For each internal photon line, with μ and ν given by the vertices used for the charge line, give a factor of $-ig_{\mu\nu}/q^2 + i\epsilon$ if we are in the $\xi = 1$ gauge, or more generally $i(-g_{\mu\nu} + (1-\xi)q_\mu q_\nu/q^2)/(q^2 + i\epsilon)$.

Multiplying this all together gives $i\mathcal{M}$.

Whew! well, its really not so bad.