Physics 613 Lecture 11 March 4, 2014

Electromagnetism; P and C

In Lecture 4 we saw that the Lagrangian density which gives Maxwell's equations for the electromagnetic fields interacting with charges and currents is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - A_\nu J^\nu.$$

The dynamical fields are A_μ, in terms of which

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

The Euler-Lagrange equation one gets by varying $\mathcal{L}$ with respect to A_ν are

$$\partial_\mu F^{\mu\nu} = J^\nu,$$

where J^ν is a conserved current dependent on the "matter fields", which is is to say the charged particles, whether complex scalars or Dirac particles.

One serious complication in quantizing the A^ν field is the invariance under gauge transformations. If we first consider the pure Maxwell lagrangian without the matter fields, we see $F^{\mu\nu}$ is completely unaffected by a gauge transformation under which

$$A_\nu \to A'_\nu = A_\nu + \partial_\nu \chi \tag{1}$$

for an arbitrary differentiable function $\chi(x^\mu)$. As the free lagrangian depends only on $F^{\mu\nu}$ and not on A_ν directly, the equations of motion can do nothing to determine $\chi(x^\mu)$, and the dynamical equations are nondeterministic for A_ν.

This indeterminancy is also clear in the equations of motion for A. Consider the $J = 0$ equations,

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0.$$

This appears to be four PDE equations (index ν) in the four functions A^μ, but of course $A_\nu = \partial_\nu \chi$ satisfies them for any function χ. In momentum space, the Fourier transformed equations are $(k^2\delta^\nu_\mu - k^\nu k_\mu)\tilde{A}^\mu = 0$, and we see that contracting these four equations with k^ν gives an identity independent of $\tilde{A}^\mu$, so we really have only three equations. One last way of seeing something is wrong is to ask for the canonical momenta Π_ν conjugate to A^ν, which are

$$\Pi_\nu := \frac{\partial \mathcal{L}}{\partial \dot{A}^\nu} = F^{0\nu},$$

which shows, by the antisymmetry of F, that $\Pi_0 \equiv 0$, which is a constraint rather than an equation of motion on phase space, and also makes it impossible to have Π_0 not commute with A^0.

Even if we have an interaction with a conserved current, the gauge transformation (1) changes the action by

$$\delta S = \int d^4x\, \delta\mathcal{L} = -\int d^4x\, J^\nu\, \partial_\nu \chi = -\int d^4x\, \partial_\nu\left(\chi J^\nu\right) = -\int_S d^3S_\nu J^\nu \chi \sim 0,$$

where the third equal sign is because the current J^ν is conserved ($\partial_\nu J^\nu = 0$), the fourth is by Gauss' Law (the divergence theorem) with S a three-dimensional hypersurface surrounding all of spacetime, and the $\sim$ implying that all currents vanish at infinity. Thus variation by χ does not give an equation of motion.

Note that the lack of determinism for A_ν classically does not mean that observable physics is undetermined, because the electric and magnetic fields depend only on $F^{\mu\nu}$. In order to have deterministic equations for A^ν, one may impose an arbitrary *gauge condition*. A popular one is the Lorenz[1] gauge, $\partial_\nu A^\nu = 0$. If we start off with any field A_ν, we can find an equivalent field A' satisfying the Lorenz condition by choosing χ such that $\partial_\nu A'^\nu = 0 = \partial_\nu A^\nu + \partial_\nu\partial^\nu\chi$, by solving the four dimensional Poisson equation $\Box\chi = -\partial_\nu A^\nu$ (which we could do with a Green's function).

If we agree to impose the Lorenz condition, the equations of motion become simply the massless Klein-Gordon equation for each component, $\Box A^\mu = 0$, which means the solutions are

$$A^\mu = \epsilon^\mu e^{ik_\nu x^\nu} \quad \text{with} \quad k_0 = \pm|\vec{k}|.$$

The constant 4-vector ϵ^μ is the *polarization vector*, and shows that there are four components, although the Lorenz condition constrains it with $k_\mu \epsilon^\mu = 0$.

[1]The book wrongly attributes this to Hendrik Antoon Lorentz, the guy with the invariance, but it is actually due to Ludvig Lorenz.

Actually the Lorenz condition does not uniquely determine the gauge, because if χ is a solution of the Laplace equation, $\Box\chi = 0$, a further gauge transformation $A_\nu \to A'_\nu = A_\nu + \partial_\nu\chi$ leaves $\partial^\nu A'_\nu = \partial^\nu A_\nu + \partial^\nu\partial_\nu\chi = 0+0 = 0$ unchanged. This means $\epsilon^\mu \to \epsilon^\mu + \beta k^\mu$ leaves the physics invariant (as $k^2 = 0$ for solutions). These complicated considerations tell us that although A is a four-component vector, a physical photon has only two polarizations. The component parallel to k^μ (in four dimensions) has no effect, and only components perpendicular to k^μ, in the sense that $k^\mu\epsilon_\mu = 0$, are allowed. Note for a photon moving in the z direction, $k^\mu = (E, 0, 0, E)$, the physical polarizations are superpositions of $\epsilon^\mu(\lambda = \pm 1) = \mp\frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$, where λ is the helicity (and these are each circularly polarized). For photons in other directions, the polarization vectors are suitably rotated. We have normalized the polarizations so that $\epsilon^*_\mu(\lambda)\epsilon^\mu(\lambda') = -\delta_{\lambda\lambda'}$.

Thus we see that the most general solution of the free Maxwell equations is

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2|\vec{k}|}} \sum_\lambda \left[\epsilon^\mu(\vec{k},\lambda)\alpha(\vec{k},\lambda)e^{-ik_\nu x^\nu} + \epsilon^{\mu\,*}(\vec{k},\lambda)\alpha^*(\vec{k},\lambda)e^{ik_\nu x^\nu}\right]\Big|_{k^0=|\vec{k}|}.$$

Quantization

The various presentations of the gauge problem all manifest themselves when it comes to trying to quantize the A_μ field. The fact that there is not even an expression $\Pi^0(A_\nu, \dot{A}_\nu)$ other than 0 means canonical quantization is impossible. For the scalar and Dirac fields we saw the propagator is the Green's function for the equation of motion, or in momentum space just the inverse, but the equation of motion for $\tilde{A}_\nu(k)$, $(-k^2\delta^\nu_\mu + k_\mu k^\nu)\tilde{A}_\nu = M_\mu{}^\nu(k)\tilde{A}_\nu = 0$ does not have an inverse as $M_\mu{}^\nu(k)k_\nu = 0$ shows M is a singular matrix.

These problems would have been ameliorated if the Lagrangian had been

$$\mathcal{L}_\xi = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2,$$

where ξ is a constant known as the *gauge parameter*. Then

$$\frac{\partial\mathcal{L}_\xi}{\partial\partial_\mu A^\nu} = F_\nu{}^\mu - \frac{1}{\xi}\delta^\mu_\nu\left(\partial_\rho A^\rho\right),$$

so the equations of motion are

$$\partial_\mu\left(\partial^\mu A_\nu - \partial_\nu A^\mu\right) + \frac{1}{\xi}\partial_\nu\partial_\rho A^\rho = 0.$$

In momentum space this is $M_\xi(k)_\mu{}^\nu\tilde{A}_\nu = 0$, with $M_\xi(k)_\mu{}^\nu = -k^2\delta^\nu_\mu + \dfrac{\xi-1}{\xi}k_\mu k^\nu$, and we see the inverse

$$\left(M_\xi^{-1}(k)\right)^{\rho\mu} = \frac{-g^{\rho\mu} + (1-\xi)k^\rho k^\mu/k^2}{k^2 + i\epsilon}$$

exists, and provides $iM^{-1}(k)$ as the propagator.

Note also that with the ξ lagrangian and equations of motion,

$$k^\mu M_\xi(k)_\mu{}^\nu\tilde{A}_\nu = 0 = \left(k^2k^\nu + \frac{\xi-1}{\xi}k^2k^\nu\right)\tilde{A}_\nu$$

so either $k^2 = 0$ or $k^\nu\tilde{A}_\nu = 0$, but furthermore, if $k^\nu\tilde{A}_\nu = 0$, $M_\xi(k)_\mu{}^\nu\tilde{A}_\nu = -k^2\tilde{A}_\mu = 0$ so we still have $k^2 = 0$ for any nonvanishing A, and if we have $k^2 = 0$, $M_\xi(k)_\mu{}^\nu\tilde{A}_\nu = \frac{\xi-1}{\xi}k_\mu k^\nu\tilde{A}_\nu = 0$ so we still have $k^\nu\tilde{A}_\nu = 0$ unless $\xi = 1$. Thus the ξ lagrangian for $0 \neq \xi \neq 1$ gives both the Lorenz condition and $\Box A = 0$ as equations of motion, which also imply the Maxwell equations.

What can this all mean? We have a continuum of different lagrangians all of which give Maxwell's equations, one of which ($\xi = 0$) has gauge invariance and the others give deterministic equations which can be canonically quantized. What we will do is take the quantization from the $\xi = 1$ case, which takes $\Box A^\nu = 0$ as the equation of motion for all four components, and we will impose the Lorenz condition on all external photons. It will turn out, because of the Ward identity, that the arbitrary piece $\propto (1-\xi)k^\mu k^\nu$ in the propagator will turn out not to give any contribution. This can only become clearer when we discuss interactions.

Electromagnetic Interactions

As we have discussed several times already, the way to incorporate electromagnetism into the dynamics of charged fields is by minimal substitution, replacing all partial derivative ∂_μ acting on a field of charge q with a covariant derivative $\hat{D}_\mu = \partial_\mu + iq\hat{A}_\mu$. Thus the Dirac lagrangian becomes $\mathcal{L}_q = \hat{\bar{\psi}}\left(i\gamma^\mu\hat{D}_\mu - m\right)\hat{\psi} = \mathcal{L}_{\rm Dirac} - q\hat{\bar{\psi}}\gamma^\mu\hat{\psi}\hat{A}_\mu$, where $\mathcal{L}_{\rm Dirac}$ is the free Dirac lagrangian density. Of course the full lagrangian density will also have to include that for free electromagnetism, $\mathcal{L}_{\rm Maxwell}$. Note that our interaction term $\hat{\mathcal{L}}_{\rm int} = -q\hat{\bar{\psi}}\gamma^\mu\hat{\psi}\hat{A}_\mu$ does not contain derivatives of any of the fields, so it doesn't change the canonical momenta or the quantization conditions.

The interaction hamiltonian density is $\hat{\mathcal{H}}' = \hat{J}_\mu \hat{A}^\mu$, where $J^\mu = q\hat{\bar{\psi}}\gamma^\mu\hat{\psi}$ now includes a factor of the charge and is the conserved electromagnetic current.

We will be able to do perturbation theory as we did for real scalar fields, with the appropriate propagators and vertices. As we have charged particles and propagators which are time-ordered vacuum expectation values of a field and its hermitian conjugate, the lines have a direction, drawn in the direction of particle flow (or opposite anti-particle flow), which is not necessarily in a direction forward in time. Also, as the Dirac fields have Dirac indices, the Dirac propagators are matrices in spinor space, and the photon propagators have Lorentz indices.

P, C and T for Quantum Fields

As we have seen, symmetries R act on quantum operators $\hat{\mathcal{O}}$ by conjugation, $\hat{\mathcal{O}}_R = \hat{R}\hat{\mathcal{O}}\hat{R}^{-1}$. As our quantum fields are operators, the possible symmetries P, C and T will give transformed fields. So $\hat{\phi}_P(\vec{x},t) = \hat{P}\hat{\phi}(\vec{x},t)\hat{P}^{-1}$, $\hat{\psi}_P(\vec{x},t) = \hat{P}\hat{\psi}(\vec{x},t)\hat{P}^{-1}$, *etc.*. But we have already discussed what the relation between ϕ_P and ϕ, and between ψ_P and ψ, should be, so we expect

$$\begin{aligned} \hat{\phi}_P(\vec{x},t) &= \hat{\phi}(-\vec{x},t), & \hat{\vec{A}}_P(\vec{x},t) &= -\hat{\vec{A}}(-\vec{x},t), \\ \hat{\psi}_P(\vec{x},t) &= \gamma^0\hat{\psi}(-\vec{x},t), & \hat{A}^0_P(\vec{x},t) &= \hat{A}^0(-\vec{x},t) \end{aligned}$$

where we have made simple choices for the arbitrary phase factors.

To find how these discrete symmetries act on the creation and annihilation operators, let us first take the simple case of the real scalar field.

$$\begin{aligned} \hat{\phi}_P(\vec{x},t) &= \hat{P}\hat{\phi}(\vec{x},t)\hat{P}^{-1} \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}} \left[\hat{P}\hat{a}(\vec{k})\hat{P}^{-1}e^{-i\omega t+i\vec{k}\cdot\vec{x}} + \hat{P}\hat{a}^\dagger(\vec{k})\hat{P}^{-1}e^{i\omega t-i\vec{k}\cdot\vec{x}}\right] \\ &= \hat{\phi}(-\vec{x},t) = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}} \left[\hat{a}(\vec{k})e^{-i\omega t-i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger(\vec{k})e^{i\omega t+i\vec{k}\cdot\vec{x}}\right] \end{aligned}$$

We can make the exponential factors line up if we replace the integration variable $\vec{k} \to -\vec{k}$ in the last expression, so we see

$$\hat{P}\hat{a}(\vec{k})\hat{P}^{-1} = \hat{a}(-\vec{k}), \quad \hat{P}\hat{a}^\dagger(\vec{k})\hat{P}^{-1} = \hat{a}^\dagger(-\vec{k}).$$

That was simple enough. Now consider the Dirac field

$$\begin{aligned} \hat{\psi}_P(\vec{x},t) &= \hat{P}\hat{\psi}(\vec{x},t)\hat{P}^{-1} = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}} \sum_s \\ &\qquad \left[\hat{P}\hat{c}_s(\vec{k})\hat{P}^{-1}u(\vec{k},s)e^{-i\omega t+i\vec{k}\cdot\vec{x}} + \hat{P}\hat{d}^\dagger_s(\vec{k})\hat{P}^{-1}v(\vec{k},s)e^{i\omega t-i\vec{k}\cdot\vec{x}}\right] \\ &= \gamma^0\hat{\psi}(-\vec{x},t) = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}} \sum_s \\ &\qquad \left[\hat{c}_s(\vec{k})\gamma^0u(\vec{k},s)e^{-i\omega t-i\vec{k}\cdot\vec{x}} + \hat{d}^\dagger_s(\vec{k})\gamma^0v(\vec{k},s)e^{i\omega t+i\vec{k}\cdot\vec{x}}\right] \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}} \sum_s \\ &\qquad \left[\hat{c}_s(-\vec{k})\gamma^0u(-\vec{k},s)e^{-i\omega t+i\vec{k}\cdot\vec{x}} + \hat{d}^\dagger_s(-\vec{k})\gamma^0v(-\vec{k},s)e^{i\omega t-i\vec{k}\cdot\vec{x}}\right] \end{aligned}$$

Once again we have reversed the integration variable $\vec{k}$ to get the exponentials to agree, and we see that

$$\hat{P}\hat{c}_s(\vec{k})\hat{P}^{-1} = \hat{c}_s(-\vec{k}), \quad \hat{P}\hat{d}_s(\vec{k})\hat{P}^{-1} = -\hat{d}_s(-\vec{k}),$$

because γ^0 reverses the signs of the lower two components of u and v, which together with $\vec{p} \to -\vec{p}$ leaves u unchanged but reverses the sign of v,

$$\gamma^0u(-\vec{k},s) = u(\vec{k},s), \qquad \gamma^0v(-\vec{k},s) = -v(\vec{k},s).$$

We assume P is a unitary operator[2] so from $\hat{\psi}_P(\vec{x},t) = \hat{P}\hat{\psi}(\vec{x},t)\hat{P}^{-1}$, we have

$$\begin{aligned} \hat{\psi}^\dagger_P(\vec{x},t) &= \left(\hat{P}\hat{\psi}(\vec{x},t)\hat{P}^{-1}\right)^\dagger = \hat{P}\hat{\psi}^\dagger(\vec{x},t)\hat{P}^{-1} \\ &= \left(\gamma^0\psi(-\vec{x},t)\right)^\dagger = \psi^\dagger(-\vec{x},t)\gamma^0. \end{aligned}$$

Then under parity, a bilinear with an arbitrary spinor matrix Γ, will be transformed

$$\hat{\bar{\psi}}(\vec{x},t)\Gamma\hat{\psi}(\vec{x},t) \underset{P}{\longrightarrow} \left(\hat{\bar{\psi}}(\vec{x},t)\Gamma\hat{\psi}(\vec{x},t)\right)_P = \hat{\psi}^\dagger(-\vec{x},t)\gamma^0\Gamma\gamma^0\hat{\psi}(-\vec{x},t)$$

Any arbitrary spinor matrix can be expanded in terms of γ^μ's, and we see that each γ^j will cause a change of sign due to anticommuting with γ^0, but γ^0's will

[2] We will have trouble with T, as we shall see, but P and C can be implemented as unitary operators.

not. In particular, our electromagnetic current $\hat{J}^\mu(\vec{x},t) = q\hat{\bar{\psi}}(\vec{x},t)\gamma^\mu\hat{\psi}(\vec{x},t)$ will transform as a polar vector should,

$$\hat{\vec{J}}_P(\vec{x},t) = -\hat{\vec{J}}(-\vec{x},t), \qquad \hat{J}^0_P(\vec{x},t) = +\hat{J}^0(-\vec{x},t).$$

As this is also the way A^μ transforms, we see that the interaction term $-\int d^4x\, A^\mu(x)J_\mu(x)$ is invariant under parity. The other bilinears also transform as we saw in Lecture 5, with $\hat{\bar{\psi}}\hat{\psi}$ a scalar, $i\hat{\bar{\psi}}\gamma_5\hat{\psi}$ a pseudoscalar, $\hat{\bar{\psi}}\gamma^\mu\gamma_5\hat{\psi}$ an axial vector, and $\frac{i}{2}\hat{\bar{\psi}}[\gamma^\mu,\gamma^\nu]\hat{\psi}$ a tensor.

Under charge conjugation, particles are supposed to be interchanged with antiparticles with no change in x^μ or p^μ. Note antiparticles still have positive energy. Thus we expect

$$\hat{C}\hat{\phi}(\vec{x},t)\hat{C}^{-1} = \hat{\phi}^\dagger(\vec{x},t), \qquad \hat{C}\hat{\psi}(\vec{x},t)\hat{C}^{-1} = i\gamma^2(\hat{\psi}^\dagger)^T(\vec{x},t),$$
$$\hat{C}\hat{A}^\mu(\vec{x},t)\hat{C}^{-1} = -iA^\mu(\vec{x},t),$$

where we have used our argument from the Dirac picture for how the components of ψ mix. We also need the transpose to keep $\hat{\psi}_P$ a column spinor rather than a row.

Expanding the fields once again tells us how the creation and annihilation operators transform. For complex scalars, $\hat{C}\hat{a}(\vec{k})\hat{C}^{-1} = \hat{b}(\vec{k})$, $\hat{C}\hat{a}^\dagger(\vec{k})\hat{C}^{-1} = \hat{b}^\dagger(\vec{k})$, and similarly with a and b interchanged. Similarly for the Dirac field, $\hat{C}\hat{c}_s(\vec{k})\hat{C}^{-1} = \hat{d}_s(\vec{k})$, $\hat{C}\hat{c}_s^\dagger(\vec{k})\hat{C}^{-1} = \hat{d}_s^\dagger(\vec{k})$, and similarly with c and d interchanged. For this to work, we need to have

$$\begin{aligned}\hat{C}\hat{\psi}\hat{C}^{-1} &= \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}}\sum_s\left[\hat{d}_s(\vec{k})u(\vec{k},s)e^{-ik_\mu x^\mu} + \hat{c}_s^\dagger(\vec{k})v(\vec{k},s)e^{ik_\mu x^\mu}\right]\\ &= \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}}\sum_s\left[\hat{c}_s^\dagger(\vec{k})i\gamma^2u^*(\vec{k},s)e^{ik_\mu x^\mu} + \hat{d}_s(\vec{k})i\gamma^2v^*(\vec{k},s)e^{-ik_\mu x^\mu}\right]\end{aligned}$$

which requires $i\gamma^2u^*(\vec{k},s) = v(\vec{k},s)$, $i\gamma^2v^*(\vec{k},s) = u(\vec{k},s)$. Now $i\gamma^2 = \begin{pmatrix}0 & i\sigma_2\\ -i\sigma_2 & 0\end{pmatrix}$ and $i\sigma_2(\vec{\sigma}\cdot\vec{p})^* = -\vec{\sigma}\cdot\vec{p}\, i\sigma_2$, so with

$$u(\vec{k},s) = \sqrt{E+m}\begin{pmatrix}\phi^r\\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\phi^s\end{pmatrix}, \qquad v(\vec{p},s) = \sqrt{E+m}\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi^s\\ \chi^s\end{pmatrix},$$
$$i\gamma^2v^*(\vec{k},s) = \sqrt{E+m}\begin{pmatrix}i\sigma_2\chi^{s\,*}\\ -i\sigma_2\frac{\vec{\sigma}^*\cdot\vec{k}}{E+m}\chi^{s\,*}\end{pmatrix} = \sqrt{E+m}\begin{pmatrix}i\sigma_2\chi^{s\,*}\\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m}i\sigma_2\chi^{s\,*}\end{pmatrix} = u(\vec{k},s')$$
$$i\gamma^2u^*(\vec{k},s) = \sqrt{E+m}\begin{pmatrix}i\sigma_2\frac{\vec{\sigma}^*\cdot\vec{k}}{E+m}\phi^{s\,*}\\ -i\sigma_2\phi^{s\,*}\end{pmatrix} = \sqrt{E+m}\begin{pmatrix}-\frac{\vec{\sigma}\cdot\vec{k}}{E+m}i\sigma_2\phi^{s\,*}\\ -i\sigma_2\phi^{s\,*}\end{pmatrix} = v(\vec{k},s')$$

if we set $\chi^{s'} = -i\sigma_2\phi^{s\,*}$, $\phi^{s'} = i\sigma_2\chi^{s\,*}$. If you apply C twice, this sets $\phi^{s''} = i\sigma_2\chi^{s'\,*} = i\sigma_2 i\sigma_2^*\phi^s = \phi^s$.

Note there was a certain ambiguity in our definitions of $\hat{c}_s$ because we didn't actually specify ϕ^s and χ^s, and suggested they could be chosen independently. In the Dirac picture, these actually gave the amplitudes for each spin, but in the field theory it is more appropriate to think of that amplitude as being in the $c_s^\dagger$ and $d_s^\dagger$, and the ϕ^s and χ^s as being basis vectors in spinor space. We could choose them to be what C suggests[3], with $\phi^s = i\sigma_2\chi^{s\,*}$. Only after specifying these elements can we ask if a Dirac field can be its own antiparticle. Such a field is called a Majorana spinor, and it can be be expanded as

$$\hat{\psi}_M(x) = \int d^3\vec{k}(2\pi)^3\sqrt{2\omega_k}\sum_s\left[\hat{c}_s(\vec{k})u(\vec{k},s)e^{-ik_\rho x^\rho} + \hat{c}_s^\dagger(\vec{k})v(\vec{k},s)e^{-ik_\rho x^\rho}\right].$$

the lagrangian looks like the Dirac one, $\mathcal{L} = \hat{\bar{\psi}}(i\,\partial\!\!\!/ - m)\hat{\psi}$, but here $\hat{\bar{\psi}}$ cannot be considered independent of $\hat{\psi}$, as $\hat{\psi}_{MC} = \hat{\psi}_M$, so $\hat{\bar{\psi}}_M = -i\hat{\psi}_M^T\gamma^2\gamma^0$, and the mass term $-m\hat{\bar{\psi}}_M\hat{\psi}_M = -m\hat{\psi}_M^T(-i\gamma^2\gamma^0)\hat{\psi}_M = -m\hat{\chi}^Ti\sigma_2\chi +$ hermitean conjugate.

[3]The book and I disagree with what $i\gamma^2$ is, but the book is inconsistent (perhaps) if J.8 means $\gamma^2 = \begin{pmatrix}0 & \sigma_2\\ -\sigma_2 & 0\end{pmatrix}$.