Physics 613 Lecture 8 Feb. 20, 2014

Last time we agreed to do all our quantum mechanics in the interaction picture, where the states evolve only under the perturbative part $\hat{H}'_I$ of the hamiltonian $\hat{H}$, while the operators evolve under the unperturbed hamiltonian $\hat{H}_0$. That is,

$$i\frac{d}{dt}\left|\psi(t)\right\rangle_I = \hat{H}'_I \left|\psi(t)\right\rangle_I, \qquad \hat{\mathcal{O}}_I(t) = e^{i\hat{H}_0 t}\mathcal{O}(0)e^{-i\hat{H}_0 t},$$

where the last expression holds in particular to define $\hat{H}'_I(t)$. We saw (following the book) that if we have a state $|\psi(t)\rangle$ which in the distant past (before the beams collided) was in the state $|i\rangle = |\phi(-\infty)\rangle_I \propto a^\dagger(\vec{k}_A)a^\dagger(\vec{k}_B)\,|0\rangle$, it will evolve into the state $|\phi(+\infty)\rangle_I = \hat{S}\,|\phi(-\infty)\rangle_I$ in the distant future (*i.e.* when it passes through the detectors). Detectors measure an angle and momentum, so basically they click if the final state is $|f\rangle \propto \left(\prod_f a^\dagger(p_f)\right)|0\rangle$. The amplitude to be in that state is $\langle f|\psi(\infty)\rangle_I = \langle f|\,\hat{S}\,|i\rangle$. And we learned last time that

$$\hat{S} = T\exp\left\{-i\int d^4x\,\hat{\mathcal{H}}_I(x)\right\}. \tag{1}$$

1 First Perturbative Calculations

We will be following the book in its presentation of 'ABC' theory, but first I want to clarify that the implication that this removes the problem of ϕ^3 is wrong.

The ABC theory consists of three distinguishable fields $\hat{\phi}_i$, $i = A, B, C$, each a real scalar field of mass m_i with a Klein-Gordon lagrangian, together with an interation term $-g\hat{\phi}_A\hat{\phi}_B\hat{\phi}_C$, which means there will be an interaction hamiltonian

$$\hat{H}' = g\int d^3x\,\hat{\phi}_A\hat{\phi}_B\hat{\phi}_C.$$

Do not suppose that the mass term of each hamiltonian, $+\frac{1}{2}m_i^2\hat{\phi}_i^2$ will protect us from negative energy states. While for each field with fixed values of the other two, the energy is bounded from below, the configuration where each of the fields becomes large and negative still produces states of arbitrary negative energy. As for ϕ^3 theory, this can be fixed by adding higher order terms, say $\sum \phi_i^4$, but we will not worry about this, as we are only interested in perturbation theory calculations, and our serious work, once we gather familiarity with the methods of perturbative field theory, will be on other theories that do not have this problem.

We have so far not discussed the normalization of states, except in the homework assignment for next week. There you will show that if we want Lorentz invariant measures, it is best not to use nonrelativistic normalization $\left\langle \vec{k} \middle| \vec{k}' \right\rangle = \delta^3(\vec{k} - \vec{k}\,')$, which implies completeness $\int d^3k \left|\vec{k}\right\rangle \left\langle \vec{k}\right| = 1$, but rather to normalize to

$$\left\langle \vec{k} \middle| \vec{k}' \right\rangle = 2\omega_k (2\pi)^3 \delta^3(\vec{k} - \vec{k}\,'), \qquad \text{with} \quad \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left|\vec{k}\right\rangle \left\langle \vec{k}\right| = 1\!\!1, \tag{2}$$

where $1\!\!1$ is really only the identity in the one-particle subspace.

A detailed and very thorough treatment of how scattering amplitudes are related to the matrix elements S_{fi} is given in Peskin and Schroeder, and I am not going to go through it here. The result of this discussion is that scattering is given by

$$\begin{aligned} \langle \vec{p}_1, \cdots \vec{p}_n | \, \hat{S} - 1\!\!1 \left| \vec{k}_A, \vec{k}_B \right\rangle &= (2\pi)^4 \, i \, \delta^4 \left(k_A^\mu + k_B^\mu - \sum_{j=1}^n p_j^\mu \right) \\ &\qquad \times \mathcal{M}(k_A, k_B \to \{p_j\}) \end{aligned} \tag{3}$$

where $\mathcal{M}$ is known as the invariant scattering amplitude. It contains the dynamical part of the scattering cross section, and is a Lorentz invariant function. Its square gets multiplied and integrated with a *relativistically invariant n-body phase space* factor to give the cross section

$$\begin{aligned} d\sigma &= \frac{1}{2E_A \, 2E_B \, |\vec{v}_A - \vec{v}_B|} \left(\prod_{f=1}^n \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\qquad \times (2\pi)^4 \, \delta^4 \left(k_A^\mu + k_B^\mu - \sum_{j=1}^n p_j^\mu \right) |\mathcal{M}(k_A, k_B \to \{p_f\})|^2 . \end{aligned} \tag{4}$$

The invariant amplitude also gives the partial decay width of a particle A at rest

$$\begin{aligned} d\Gamma &= \frac{1}{2m_A} \left(\prod_{f=1}^n \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\qquad \times (2\pi)^4 \, \delta^4 \left(k_A^\mu - \sum_{f=1}^n p_f^\mu \right) \left| \mathcal{M}(m_A \to \{p_f\}) \right|^2 . \end{aligned} \tag{5}$$

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The $\mathcal{A}^{(1)}_{fi}$ of 6.48 is just the lowest order contribution to $\hat{S}-1$, so from (3) we have $\mathcal{M}=-g$. For the total decay width, we need to integrate over the final momenta, so

$$\begin{aligned}\Gamma &= \int \frac{d^3p_A\, d^3p_B}{(2\pi)^6\, 8M_C E_A E_B}(2\pi)^4\delta^3(\vec{p}_A+\vec{p}_B)\delta(E_A+E_B-m_C)(-g)^2 \\ &= \frac{g^2}{8\pi M_C}\int_0^\infty p^2 dp \frac{\delta(E_A+E_B-m_C)}{E_A\, E_B}\end{aligned}$$

The particles emerge with $|\vec{p}|$ the solution to $\sqrt{p^2+m_A^2}+\sqrt{p^2+m_B^2}=m_C$, which is correctly given in 6.65. Using the general formula

$$\int dx\, g(x)\, \delta(f(x)) = \sum_{f(x_j)=0} \frac{g(x_j)}{|f'(x_j)|}$$

with $g(p)=p^2/E_A E_B$, $f(p)=\sqrt{p^2+M_A^2}+\sqrt{p^2+M_B^2}-m_C$, so $f'(p)=\dfrac{p}{E_A}+\dfrac{p}{E_B}$, we have

$$\Gamma = \frac{g^2}{8\pi M_C}\frac{p^2}{E_A\, E_B}\frac{1}{\frac{p}{E_A}+\frac{p}{E_B}} = \frac{g^2}{8\pi m_C}\frac{p^2}{p(E_B+E_A)} = \frac{g^2}{8\pi M_C^2}p.$$

Notice that as Γ is a decay width (in energy) or the rate of decay, which has units of $\sec^{-1}\equiv \mathrm{m}^{-1}\equiv$ mass (because we have taken $c=1$ and $\hbar=1$), g must have dimensions of mass.

Let's say a few words about dimensional analysis. Because we are taking $c=1$ and $\hbar=1$, the only units left can be expressed as powers of mass. Lengths and times are mass^{-1}, and derivatives with respect to x^μ have dimensions of mass. The arguments of exponentials must always be dimensionless, so the action must be, and as it is the four-dimensional integral of the lagrangian density, $\mathcal{L}$ must have dimension mass^4. Each term in the Klein-Gordon $\mathcal{L}=\frac{1}{2}\left[\left(\partial_\mu\hat{\phi}\right)\partial^\mu\hat{\phi}-m^2\hat{\phi}^2\right]$ then shows us $\hat{\phi}$ must have dimension mass^1. The Maxwell $\mathcal{L}=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ tells us $F_{\mu\nu}$ has dimension mass^2, and as F is given by single derivatives of A_ν, A_ν has dimension mass^1 as well.

As we added a term $-g\hat{\phi}_A\hat{\phi}_B\hat{\phi}_C$ to the Lagrangian, and as each $\hat{\phi}$ already has dimension mass^1, the constant g must also have dimension mass^1, consistent with what we just argued from the width.

Note that the covariant derivative $D_\mu = \partial_\mu + iqA_\mu$ must also have dimension 1, so the electric charge q is dimensionless. If we put back the relevant powers of $\hbar$ and c, we can define the dimensionless coupling constant of electromagnetism, known as the *fine structure constant*

$$\alpha = \frac{e^2}{4\pi\hbar c}$$

in our units, or $e^2/4\pi\epsilon_0\hbar c$ in SI units.

1.1 Scattering: $A + B \to A + B$

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Let me attack equation 6.74 a bit differently than the book does. First, let's undo the trick that let us integrate over all t_1 and t_2, and write

$$\begin{aligned}
\langle \vec{p}\,'_A \vec{p}\,'_B | \, \hat{S} \, | \vec{p}_A \vec{p}_B \rangle \;\; &\approx \;\; (-ig)^2 \sqrt{16 E_A E'_A E_B E'_B} \iint_{t_1 > t_2} d^4x_1 d^4x_2 \; \langle 0 | \, \hat{a}_A(p'_A) \hat{a}_B(p'_B) \\
&\qquad \hat{\phi}_A(x_1)\hat{\phi}_B(x_1)\hat{\phi}_C(x_1)\hat{\phi}_A(x_2)\hat{\phi}_B(x_2)\hat{\phi}_C(x_2)\hat{a}^\dagger_A(p_A)\hat{a}^\dagger_B(p_B) \, |0\rangle \\
&= (-ig)^2 \sqrt{16 E_A E'_A E_B E'_B} \iint_{t_1 > t_2} d^4x_1 d^4x_2 \\
&\qquad \langle 0 | \hat{a}_A(p'_A)\hat{\phi}_A(x_1)\hat{\phi}_A(x_2)\hat{a}^\dagger_A(p_A)|0\rangle \\
&\qquad \langle 0 | \, \hat{a}_B(p'_B)\hat{\phi}_B(x_1)\hat{\phi}_B(x_2)\hat{a}^\dagger_B(p_B) \, |0\rangle \, \langle 0 | \, \hat{\phi}_C(x_1)\hat{\phi}_C(x_2) \, |0\rangle
\end{aligned}$$

The last factor is simple, as the annihilation piece of $\hat{\phi}_C(x_2)$ vanishes acting on $|0\rangle$, and the creation piece of $\langle 0| \, \hat{\phi}_C(x_1)$ vanishes as well, so we have only

$$\begin{aligned}
\langle 0 | \, \hat{\phi}_C(x_1)\hat{\phi}_C(x_2) \, |0\rangle \; &= \; \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \, e^{-ik\cdot x_1} \int \frac{d^3k'}{(2\pi)^3\sqrt{2\omega_k}} \, e^{ik'\cdot x_2} \, \langle 0 | \, \hat{a}(k)\hat{a}^\dagger(k') \, |0\rangle \\
&= \; \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik_\mu(x_1-x_2)^\mu}.
\end{aligned}$$

We will call this $D(x_1, x_2)$ or $D(x_1 - x_2)$ when the relevant mass is obvious, but here we will need to call it $D(x_1 - x_2, m_C)$, because the relevant $\omega_k = \sqrt{\vec{k}^{\,2} + m_C^2}$ here.

The factors involving A or B fields are somewhat more complicated. In $\langle 0| \, \hat{a}_A(p'_A)\hat{\phi}_A(x_1)\hat{\phi}_A(x_2)\hat{a}^\dagger_A(p_A) \, |0\rangle$ we must get one creation operator and one annihilation operator from the two $\hat{\phi}$'s, as each creation operator needs to be contracted with an annihilation piece before it reaches the $\langle 0|$. That is, as we commute the $\hat{a}_A^\dagger(p_A)$ to the left, we might pick up the commutator with one of the $\hat{\phi}$'s, giving us a factor

$$[\hat{\phi}(x), \hat{a}_A^\dagger(p_A)] = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} e^{-ik\cdot x}[\hat{a}(k), \hat{a}_A^\dagger(p_A)] = \frac{1}{\sqrt{2\omega_{P_A}}} e^{-ip_A\cdot x},$$

or it could contract with the $\hat{a}_A(p'_A)$ giving $(2\pi)^3\delta^3(p'_A - p_A)$. In the second case this would leave $\langle 0|\,\hat{\phi}_A(x_1)\hat{\phi}_A(x_2)\,|0\rangle = D(x_1 - x_2)$, so this case gives a contribution $(2\pi)^3\delta^3(p'_A - p_A)\,D(x_1 - x_2)$ to the particle A factor. Returning to the first case, we are left with $\langle 0|\,\hat{a}_A^\dagger(p'_A)\hat{\phi}_A(x')\,|0\rangle = \frac{1}{\sqrt{2\omega_{p'_A}}} e^{ip'_A\cdot x'}$, so combined we get $\frac{1}{2\sqrt{\omega_{p'_A}\omega_{p_A}}} e^{ip'_A\cdot x' - ip_A\cdot x}$. In this case we have both $x = x_1$, $x' = x_2$ and vice versa, so the total expression from the A fields is

$$\frac{1}{2\sqrt{\omega_{p_A}\omega_{p'_A}}}\left(e^{ip'_A\cdot x_1 - ip_A\cdot x_2} + e^{ip'_A\cdot x_2 - ip_A\cdot x_1}\right) + (2\pi)^3\delta^3(p'_A - p_A)\,D(x_1 - x_2, m_A).$$

Of course the same calculation applies to the B field piece.

Let us change variables, defining $y^\mu = x_1^\mu - x_2^\mu$, so the A piece is

$$\frac{1}{2\sqrt{\omega_{p_A}\omega_{p'_A}}}\left(e^{ip'_A\cdot y} + e^{-ip_A\cdot y}\right) e^{i(p'_A - p_A)\cdot x_2} + (2\pi)^3\delta^3(p'_A - p_A)\,D(y, m_A),$$

the B is

$$\frac{1}{2\sqrt{\omega_{p_B}\omega_{p'_B}}}\left(e^{ip'_B\cdot y} + e^{-ip_B\cdot y}\right) e^{i(p'_B - p_B)\cdot x_2} + (2\pi)^3\delta^3(p'_B - p_B)\,D(y, m_B),$$

and the C piece is $D(y)$. So all together,

$$\begin{aligned}
\langle \vec{p}'_A\vec{p}'_B|\,\hat{S}\,|\vec{p}_A\vec{p}_B\rangle &\approx (-ig)^2\int_0^\infty d^4y\int_{-\infty}^\infty d^4x_2\,D(y, m_C)\\
&\times\left[\left(e^{ip'_A\cdot y} + e^{-ip_A\cdot y}\right) e^{i(p'_A - p_A)\cdot x_2} + 2\sqrt{E_A E'_A}(2\pi)^3\delta^3(p'_A - p_A)\,D(y, m_A)\right]\\
&\times\left[\left(e^{ip'_B\cdot y} + e^{-ip_B\cdot y}\right) e^{i(p'_B - p_B)\cdot x_2} + 2\sqrt{E_B E'_B}(2\pi)^3\delta^3(p'_B - p_B)\,D(y, m_B)\right]
\end{aligned}$$

In multiplying out the []'s, there is one term with no exponentials and $D^3(y)\delta^3(p'_A - p_A)\delta^3(p'_B - p_B)$, with no x_2 dependence and therefore an infinite integral proportional to the volume of space and the infinite time interval from $|i\rangle$ to $\langle f|$, but with no scattering, as $\vec{p}\,'_A = \vec{p}_A$ and $\vec{p}\,'_B = \vec{p}_B$. There is also a piece with $D^2(y)\delta^3(p'_A - p_A)$ multiplying $e^{i(p'_B - p_B)\cdot x_2}$, which upon integration gives the other delta, $\vec{p}\,'_B = \vec{p}_B$. So the interesting piece has only the one $D(y, m_C)$, and it has a x_2 dependence $\exp\left[i\left(p'_B - p_B + p'_A - p_A\right)\cdot x_2\right]$ which upon integration gives $(2\pi)^4\delta\left(p'_B - p_B + p'_A - p_A\right)$. So the actual scattering amplitude is given by

$$\begin{aligned}\langle \vec{p}'_A\vec{p}'_B|\, \hat{S}\, |\vec{p}_A\vec{p}_B\rangle \;\approx\; & -g^2(2\pi)^4\delta\left(p'_B - p_B + p'_A - p_A\right)\int d^4y\, \Theta(y^0)\, D(y, m_C) \\ & \times\left[e^{i(p'_A + p'_B)\cdot y} + e^{i(p'_A - p_B)\cdot y} + e^{-i(p_A - p'_B)\cdot y} + e^{-i(p_A + p_B)\cdot y}\right]\end{aligned}$$

Notice that because of the δ^4 function, the first and fourth exponentials differ only in sign, and similarly for the second and third terms. Reversing the sign of y in the last two terms, we have

$$\begin{aligned}\langle \vec{p}'_A\vec{p}'_B|\, \hat{S}\, |\vec{p}_A\vec{p}_B\rangle \approx & -g^2(2\pi)^4\delta\left(p'_B - p_B + p'_A - p_A\right) \\ & \times \int d^4y \left[e^{i(p'_A + p'_B)\cdot y} + e^{i(p'_A - p_B)\cdot y}\right] \\ & \times \left[\Theta(y^0)\, D(y, m_C) + \Theta(-y^0)\, D(-y, m_C)\right].\end{aligned}$$

Recall that, due to the translation invariance of D,

$$\Theta(-y^0)\, D(-y) = \Theta(-y^0)\, \langle 0|\, \hat{\phi}(-y)\hat{\phi}(0)\, |0\rangle = \Theta(-y^0)\, \langle 0|\, \hat{\phi}(0)\hat{\phi}(y)\, |0\rangle\,,$$

and of course $\Theta(y^0)\, D(y) = \Theta(y^0)\, \langle 0|\, \hat{\phi}(y)\hat{\phi}(0)\, |0\rangle$ and in both cases the bracket is time ordered, $\langle 0|\, T\hat{\phi}(y)\hat{\phi}(0)\, |0\rangle$. Define the Feynman propagator $D_F(x_1 - x_2) := \langle 0|\, T\hat{\phi}(x_1)\hat{\phi}(x_2)\, |0\rangle$ and its Fourier transform as $\tilde{D}_F(q^\mu) = \int d^4x e^{iq_\mu x^\mu} D_F(x^\mu)$, and we see that the scattering part

$$\begin{aligned}\langle \vec{p}'_A\vec{p}'_B|\, \hat{S}\, |\vec{p}_A\vec{p}_B\rangle \;\approx\; & -g^2(2\pi)^4\delta\left(p'_B - p_B + p'_A - p_A\right) \\ & \left[\tilde{D}_F(p_A + p_B, m_C) + \tilde{D}_F(p'_A - p_B, m_C)\right].\end{aligned}$$