Physics 613 Lecture 7 Feb. 18, 2014

1 Free Theory States, and Interactions

We have seen that the theory of a real scalar field

$$\hat{\phi}(x^\mu) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega}} \left[\hat{a}(\vec{k})e^{-ik\cdot x} + \hat{a}^\dagger(\vec{k})e^{ik\cdot x}\right]. \tag{1}$$

is now a quantum-mechanical operator, and the Hamiltonian,

$$\hat{H} = \int \frac{d^3k}{(2\pi)^3}\, \omega_{\vec{k}}\, \hat{a}^\dagger(\vec{k})\, \hat{a}(\vec{k}) \tag{2}$$

is also, so we should ask what is the space upon which they act. Their action is all through the operators $\hat{a}(\vec{k})$ and $a^\dagger(\vec{k})$, which obey

$$[\hat{a}(\vec{k}), \hat{a}(\vec{k}\,')] = 0, \quad [\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}\,')] = (2\pi)^3\delta^3(\vec{k}-\vec{k}\,'), \quad [\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{k}\,')] = 0. \tag{3}$$

The clue is to consider the commutator of $\hat{H}$ with $\hat{a}(\vec{k})$ and $a^\dagger(\vec{k})$,

$$\begin{aligned}
[\hat{H}, \hat{a}(\vec{p})] &= \int \frac{d^3k}{(2\pi)^3}\, \omega_{\vec{k}}\, [\hat{a}^\dagger(\vec{k})\, \hat{a}(\vec{k}), \hat{a}(\vec{p})] = \int \frac{d^3k}{(2\pi)^3}\, \omega_{\vec{k}}\, [\hat{a}^\dagger(\vec{k}), \hat{a}(\vec{p})]\, \hat{a}(\vec{k}) \\
&= -\int \frac{d^3k}{(2\pi)^3}\, \omega_{\vec{k}}\, (2\pi)^3\delta^3(\vec{k}-\vec{p}\,')\, \hat{a}(\vec{k}) = -\omega_{\vec{p}}\, \hat{a}(\vec{p}), \qquad (4) \\
[\hat{H}, a^\dagger(\vec{p})] &= \omega_{\vec{p}}\, \hat{a}^\dagger(\vec{p}). \qquad (5)
\end{aligned}$$

Equation (4) has an important consequence — given a state $|\psi\rangle$ with energy E (that is, $\hat{H}\,|\psi\rangle = E\,|\psi\rangle$) there is another state $|\psi'\rangle = \hat{a}(\vec{p})\,|\psi\rangle$ with energy $E - \omega_p$, because

$$\begin{aligned}
\hat{H}\,|\psi'\rangle &= \hat{H}\hat{a}(\vec{p})\,|\psi\rangle = [\hat{H}, \hat{a}(\vec{p})]\,|\psi\rangle + \hat{a}(\vec{p})\hat{H}\,|\psi\rangle \\
&= -\omega_p\hat{a}(\vec{p})]\,|\psi\rangle + \hat{a}(\vec{p})E\,|\psi\rangle = (E-\omega_p)\hat{a}(\vec{p})]\,|\psi\rangle = (E-\omega_p)\,|\psi'\rangle\,,
\end{aligned}$$

unless, of course, $\hat{a}(\vec{p})\,|\psi\rangle = 0$. But as all ω's are positive, in fact $\omega \geq m > 0$, that means that if there is a state of lowest energy, a vacuum state we will call $|0\rangle$, it must be annihilated by $\hat{a}(\vec{p})$ for *all* $\vec{p}$.

Starting with the state $|0\rangle$, which we will normalize so $\langle 0|0\rangle = 1$, we can build up infinitely many states by applying products of $\hat{a}^\dagger(\vec{p})$'s for an arbitrary collection of $\vec{p}_j$,

$$|\vec{p}_1 \cdots \vec{p}_n\rangle \propto \prod_{j=1}^{n} \hat{a}^\dagger(\vec{p}_j)\,|0\rangle\,, \tag{6}$$

which will have $E = \sum_{j=1}^{n} \omega_{p_j}$ if we claim the energy of $|0\rangle$ is zero.

From Emmy Noether we learn that if the action is invariant for any continuous symmetry transformation, there is a conserved current and a conserved "charge" associated with that transformation. Our Lagrangian density $\mathcal{L} = \frac{1}{2}\left((\partial_\mu\phi)^2 - m^2\phi^2\right)$ is a scalar under Lorentz transformations and translations, so we must have conserved total momentum $\hat{P}^\mu$ as well as angular momenta $\hat{L}_{\mu\nu}$. Of course the "charge" associated with time translation is the hamiltonian $\hat{H}$. The corresponding things for spatial translation invariance are the spatial total momenta $\hat{\vec{P}}(t) = -\int d^3x \hat{\pi}(\vec{x},t)\vec{\nabla}\hat{\phi}(\vec{x},t)$. You will not be surprized to find this gives[1]

$$\hat{\vec{P}} = \int \frac{d^3k}{(2\pi)^3}\,\vec{k}\,\hat{a}^\dagger(\vec{k})\,\hat{a}(\vec{k}),$$

and that if a state $|\phi\rangle$ is an eigenstate $\hat{\vec{P}}\,|\phi\rangle = \vec{P}\,|\phi\rangle$, that $\hat{a}^\dagger(\vec{k})\,|\phi\rangle$ is an eigenstate with momentum $\vec{P}+\vec{k}$. Assuming, of course, that the vacuum has $\vec{P} = 0$, we see that the state (6) has $\vec{P} = \sum_{j=1}^{n} \vec{p}_j$, so it is very reasonable to think of this state as a collection of n nointeracting particles, each with a given momentum $\vec{p}_j$ and (rest) mass m.

Notice that because the $a^\dagger$ all commute with each other, the states $|\vec{p}_1 \cdots \vec{p}_n\rangle$ are independent of the order in which the $\vec{p}_j$ occur. That is, the particles obey Bose statistics, and any multiparticle wave function describing this state would need to be totally symmetric under interchange of particles.

In our previous discussion where the fields were being treated as if they were wave functions, the quantum-mechanical symmetry operators $\hat{\Lambda}$, and their generators, such as $\hat{P}^\mu$ or $\hat{L}_{\mu\nu}$, were assumed to act on ϕ or ψ as states $|\Psi\rangle$. But now that we are taking ϕ and ψ to be quantum fields, which *are operators* themselves, we may also ask how these symmetry operators act on fields. If a symmetry operator $\hat{\Lambda}$ acts on all the states of the system,

[1] with a symmetry argument that $\int d^3k\,\vec{k} = 0$

$|\Psi\rangle \underset{\hat\Lambda}{\longrightarrow} |\Psi'\rangle = \hat\Lambda\,|\Psi\rangle$, and if $\hat{\mathcal{O}}$ is another operator that actively maps each state $|\Psi\rangle \underset{\hat{\mathcal{O}}}{\longrightarrow} |\Phi\rangle = \hat{\mathcal{O}}\,|\Psi\rangle$ into another physical state, then

$$|\Phi\rangle \underset{\hat\Lambda}{\longrightarrow} |\Phi'\rangle = \hat\Lambda\,\hat{\mathcal{O}}\,|\Psi\rangle = \hat{\mathcal{O}}'\,|\Psi'\rangle\,, \quad \text{where} \quad \hat{\mathcal{O}}' = \hat\Lambda\,\hat{\mathcal{O}}\,\hat\Lambda^{-1}.$$

Thus we see that in general, an operator is transformed by *conjugating* it with the symmetry operator. If this finite symmetry operator is the exponential of a generator, $\hat\Lambda = e^{-i\theta\hat{L}}$, then the way the generator acts on $\hat{\mathcal{O}}$ is by commutation, for

$$i\frac{d}{d\theta}\hat{\mathcal{O}}'(\theta)\bigg|_{\theta=0} = i\frac{d}{d\theta}\left(e^{-i\theta L}\hat{\mathcal{O}}e^{i\theta L}\right)\Big|_{\theta=0} = \left[L, \hat{\mathcal{O}}\right].$$

We have already used this for the time-translation generator $\hat{H}$ in equations (4) and (5).

2 Interactions

So far we have only quantized a theory with a lagrangian quadratic in the fields, which have linear equations of motion and correspond to noninteracting particles. But we saw from our discussion of electromagnetism interacting with charge particles, that the equations of motion, while linear in ϕ or ψ if we consider A^μ to be a fixed external field, are not really linear if we allow that A^μ is also a dynamical field, and will be affected by the charged particle fields ϕ or ψ. So the full theory, including the back reactions of the particles on the electromagnetic field, is not exactly solvable.

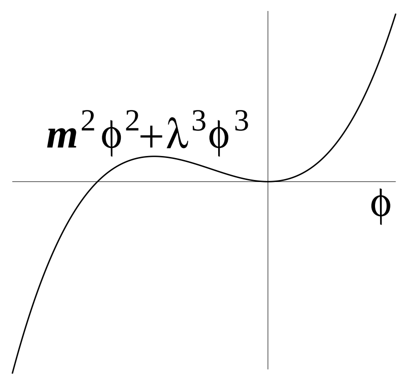
The form of the lagrangian for this interacting theory, as for example the charged scalar lagrangian from Homework 3, which includes terms like $-iq\hat{A}^\mu\hat\phi^\dagger\partial_\mu\hat\phi$, suggests that the general interaction we might wish to introduce is easy to write down, just by including terms of order greater than quadratic in the dynamical fields. We will find that all of the interactions we wish to discuss (and we exclude general relativity as usual) are of this form, cubic or quartic overall in the fields. It would be natural to therefore begin the discussion of the Lagrangian for the Dirac spinor (which is bilinear in ψ and $\bar\psi$), adding electromagnetic interactions via minimal substitutions, which would introduce a term proportional to $A^\mu\bar\psi\gamma_\mu\psi$, overall cubic in the fields. We will get to that, but the complications from the spinor nature of the Dirac field and the vector nature and gauge invariance of the electromagnetic fields introduce considerable complications we would rather postpone until later. So we will begin with a real scalar field. If we add a $\hat{\phi}^3(x)$ interaction to the Klein-Gordon lagrangian

$$\hat{\mathcal{L}}^{(\phi^3)}(x) = \frac{1}{2}(\partial_\mu \hat{\phi}(x))\partial^\mu \hat{\phi}(x) - m^2\hat{\phi}^2(x) - \lambda\hat{\phi}^3(x),$$

we see that the canonical momentum $\hat{\pi}(x) = \frac{\partial \hat{\mathcal{L}}}{\partial \dot{\hat{\phi}}} = \dot{\hat{\phi}}$ is unchanged by the interaction, and the hamiltonian is therefore

$$H = \frac{1}{2}\int d^3x \left[\hat{\pi}^2 + (\vec{\nabla}\hat{\phi})^2 + m^2\phi^2 + \lambda\hat{\phi}^3\right].$$

If we consider classical states ϕ, states which vary in time or space have higher energy, due to the $\pi^2 = \dot{\phi}^2 \geq 0$ and $(\vec{\nabla}\phi)^2 \geq 0$. States which are constant everywhere, $\phi(\vec{x}, t0 = \phi_0$ have an energy per unit volume $V(\phi_0) = \frac{1}{2}\left(m^2\phi_0^2 + \lambda\phi_0^3\right)$.



This shows that the theory isn't really well defined. Classically we would have states of arbitrarily negative energy by taking $\phi = \text{const} \to -\infty$. This could be fixed by adding a term $-\lambda_4\hat{\phi}^4(x)$ to the lagrangian, or substituting this for the $\hat{\phi}^3$ term, but that would complicate the discussion. In any case, the theory with terms higher than quadratic is not exactly solvable, and we will pursue results by using perturbation theory, assuming the λ coefficient is small and the non-quadratic terms can be treated order by order in perturbation theory. As such, we will not meet the problem of the hamiltonian being unbounded from below, though we will consider a similar situation later when we get to spontaneous symmetry breaking.

So we will divide our full hamiltonian into a free hamiltonian $\hat{H}_0 = \hat{H}_{\text{KG}}$ which is the $\lambda \to 0$ part, and an interaction hamiltonian $\hat{H}'(t) = \lambda\int d^3x\hat{\phi}^3(\vec{x}, t)$.

Recall in nonrelativistic quantum mechanics, there are two ways to view how things evolve. In the Schrödinger picture, we have fixed operators like $\hat{x}$ and $\hat{p}$, which are not considered to be functions of time, but which act on a wave function $\psi(t)$ which is time dependent, with $i\hbar\frac{d}{dt}\psi(t) = \hat{H}\psi(t)$. The wave function at any time can be formally related to that at $t = 0$ by $\psi(t) = e^{-i\hat{H}t/\hbar}\psi(0)$. The expectation value of any time-independent operator $\hat{\mathcal{O}}$ at time t is given by

$$\langle\psi(t)|\,\hat{\mathcal{O}}_{\rm Sch}\,|\psi(t)\rangle = \langle\psi(0)|\,e^{i\hat{H}t/\hbar}\,\hat{\mathcal{O}}_{\rm Sch}\,e^{-i\hat{H}t/\hbar}\,|\psi(0)\rangle$$

and so could be just as well represented by considering the state $\psi = \psi(0)$ to be static, but considering the operators $\hat{\mathcal{O}}_{\rm H}(t) = \hat{e}^{i\hat{H}t/\hbar}\hat{\mathcal{O}}_{\rm Sch}e^{-i\hat{H}t/\hbar}$ to evolve with time. This is the Heisenberg picture.

In our discussion of the field $\phi(\vec{x},t)$ we had it evolving with time. Before quantizing we could not distinguish whether it was a Schrödinger wave function or a Heisenberg operator, but now that we have quantized it, we see that $\hat{\phi}(\vec{x},t)$ was the Heisenberg operator for our noninteracting field theory. The time dependence we assigned it was consistent with the Heisenberg equation of motion,

$$\frac{d}{dt}\hat{\mathcal{O}}_{\rm H} = \frac{i}{\hbar}[\hat{H},\hat{\mathcal{O}}_{\rm H}]$$

where there is no explicit dependence of $\hat{\mathcal{O}}_{\rm Sch}$ on t. But that was only for the noninteracting hamiltonian, and as we don't know how to solve the nonlinear equations of motion for $\hat{\phi}_{\rm H}$ with the full hamiltonian, the $\hat{\phi}(\vec{x},t)$ of equation (1) is in neither the Schrödinger nor the Heisenberg picture of the full theory. Instead, we consider an intermediate formulation called the *interaction* picture.

Let's go back to setting $\hbar = 1$.

So an interaction picture for a quantum system begins by dividing the full hamiltonian into a non-interacting piece $\hat{H}_0$ and an interaction term $\hat{H}'$, assuming we know how to find the time evolution of states under $\hat{H}_0$, and that it has no explicit time dependence. Starting with the Schrödinger picture states $|\psi(t)\rangle$, define interaction-picture states $|\psi(t)\rangle_{\rm I} = e^{i\hat{H}_0t}\,|\psi(t)\rangle$. If we ask how this state evolves under the influence of the full hamiltonian, we have

$$\begin{aligned}
i\frac{d}{dt}\,|\psi(t)\rangle_{\rm I} &= \left(i\frac{d}{dt}e^{i\hat{H}_0t}\right)|\psi(t)\rangle + e^{i\hat{H}_0t}\left(i\frac{d}{dt}\,|\psi(t)\rangle\right)\\
&= e^{i\hat{H}_0t}\left(-\hat{H}_0\,|\psi(t)\rangle + i\frac{d}{dt}\,|\psi(t)\rangle\right)\\
&= e^{i\hat{H}_0t}\left(-\hat{H}_0\,|\psi(t)\rangle + (\hat{H}_0+\hat{H}')\,|\psi(t)\rangle\right)
\end{aligned}$$

$$= \quad e^{i\hat{H}_0 t}\hat{H}'\,|\psi(t)\rangle = e^{i\hat{H}_0 t}\hat{H}'e^{-i\hat{H}_0 t}\,|\psi(t)\rangle_{\rm I}\,,$$

so

$$i\frac{d}{dt}\,|\psi(t)\rangle_{\rm I} = \hat{H}'_{\rm I}\,|\psi(t)\rangle_{\rm I} \qquad \text{with} \quad \hat{H}'_{\rm I} := e^{i\hat{H}_0 t}\hat{H}'e^{-i\hat{H}_0 t}, \tag{7}$$

the interaction hamiltonian *in the interaction picture.* Notice that we started with the constant Schrödinger picture $\hat{H}'$ and transformed it only with the noninteracting hamiltonian, so all the operators involved in the expression for $\hat{H}'$ (for example $-\lambda\hat{\phi}^3$) are in fact interaction-picture operators with known time dependence.

Read pages 156-158, §6.2.2 of Aitchison and Hey, (4'th Ed.) [147-149 in 3rd Ed.?].

I will not rewrite this, as it is fine as is, except to point out

- The time-ordering symbol T is sort of a metaoperator, not an operator, because it acts on the written expression, not on the states present in the expression where it acts.
- But allowing for such metamathematics, equation 6.42 is more elegantly written as
$$\hat{S} = T\exp\left\{-i\int d^4x\,\hat{\mathcal{H}}_I(x)\right\}.$$