Physics 613 Lecture 6 Feb. 11, 2014

1 Quantum Field Theory

Today we will begin reformulating our discussion in terms of Quantum Field Theory.

1.1 Prerequisites

I am going to assume that you are familiar with

1. Lagrangian and Hamiltonian formulation of classical mechanics for discrete systems (finite number of degrees of freedom).
2. Lagrangian formulation of classical mechanics for continuous systems (fields).
3. Noether's theorem.
4. Quantum mechanics of the harmonic oscillator, Heisenberg and Schrödinger pictures, and perturbation theory.

If you need remediation on (2), please review one of

- the last chapter of Goldstein, any edition
- My 507 lecture notes, chapter 8,
`http://www.physics.rutgers.edu/∼shapiro/507/book9_2.pdf`
- My 615 lecture notes,
`http://www.physics.rutgers.edu/∼shapiro/615/lects/intro_2.pdf`

1.2 Fields

We have seen that attempts to describe relativistic mechanics in quantum mechanical terms fails because one gets negative energy states, which really means excitations are possible which excite a negative energy state into a positive energy one, which is the creation of a particle/antiparticle pair, so

that we are not dealing with a fixed number of particles or a finite number of degrees of freedom.

Relativity also requires all interactions are local, which implies we need to consider dynamics as taking place at each point in space, not globally where each particle affects every other one via a potential. So we are led to assigning a field for each kind of particle we wish to discuss.

This is already familiar for electromagnetism, where we expect dynamics to mean the dynamics of $A^\mu(\vec{x},t)$. But for this to interact with electrons, point by point in space, we must also consider the Dirac field of the electron $\psi(\vec{x},t)$ to be not an amplitude for a single particle to be at the point $\vec{x}$ but rather a dynamical degree of freedom (or several) at each point $\vec{x}$ in space.

To describe the dynamics of fields $\eta_j(\vec{x},t)$, we assume there is a given function of η_j and the first time and spatial derivatives $\eta_{j,\mu} := \partial\eta_j/\partial x^\mu$, and maybe of x^μ as well, which we call the Lagrangian density $\mathcal{L}$. Notice the x^μ are not degrees of freedom, only the values of the fields at each point are. The Lagrangian L is the spatial integral of the density, $L(t) = \int d^3x \mathcal{L}(\eta_j(x^\nu), \eta_{j,\mu}(x^\nu), x^\nu)$, but more importantly the action is the four-dimensional integral $S = \int dt L(t) = \int d^4x \mathcal{L}(\eta_j(x^\nu), \eta_{j,\mu}(x^\nu), x^\nu)$. The equations of motion are determined by insisting the variation of the action vanishes (to first order) under any variation of the fields within the four-dimensional space-time volume. This is given in terms of the variation of the lagrangian density $\mathcal{L}$ in terms of each of its arguments, so that

- $\dfrac{\partial\mathcal{L}}{\partial\eta_j(\vec{x},t)}$ means to vary the $\mathcal{L}$ function only by its dependence on η_j and not on its derivatives, and only at the one point in space-time, and
- $\dfrac{\partial\mathcal{L}}{\partial\eta_{j,\mu}}(\vec{x},t)$ means vary the $\mathcal{L}$ function only by its dependence on that derivative of η_j at only that one point, and holding η_j fixed.

Actually, as the action is an integral, the variations we need to consider will be proportional to Dirac delta functions, so we really need to express $\mathcal{L}$ as a differentiable function of η and $\eta_{,\mu}$, and take

$$\frac{\delta\eta(x_1)}{\delta\eta(x_2)} = \delta(x_1 - x_2), \quad \frac{\delta\eta_{,\mu}(x_1)}{\delta\eta(x_2)} = \frac{\delta\eta(x_1)}{\delta\eta_{,\mu}(x_2)} = 0, \quad \frac{\delta\eta_{,\mu}(x_1)}{\delta\eta_{,\nu}(x_2)} = \delta(x_1 - x_2)\delta^\nu_\mu.$$

Then it turns out the Euler-Lagrange equations of motion are

$$\frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial\eta_{j,\mu}} - \frac{\partial\mathcal{L}}{\partial\eta_j} = 0. \tag{1}$$

For each field η_j we can define a vector of fields

$$\pi_j^\mu := \frac{\partial \mathcal{L}}{\partial \eta_{j,\mu}}$$

so the Euler-Lagrange equations are also $\partial_\mu \pi_j^\mu - \frac{\partial \mathcal{L}}{\partial \eta_j} = 0$. The time component π_j^0 of π_j^μ is the canonical momentum (field) conjugate to η_j, and is sometimes called simply π_j. We may also define the hamiltonian density

$$\mathcal{H} = \sum_j \pi_j \phi_{j,0} - \mathcal{L},$$

where as usual we should replace $\phi_{j,0}$ by its value given by π and ϕ.

As an example, consider a real scalar field $\phi(x)$ with the Lagrangian density given by

$$\mathcal{L}(\phi, \phi_{,\mu}) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2.$$

We see immediately that $\pi^\mu = \partial^\mu \phi = \phi_{,\mu}$ and $\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$, so the equation of motion is

$$\partial_\mu \partial^\mu \phi - m^2 \phi = 0,$$

the Klein Gordon equation. The hamiltonian density is $\mathcal{H} = \pi\dot{\phi} - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2$.

The solutions to the equations of motion for the Klein-Gordon field are, of course, linear combinations of

$$\phi(\vec{x}, t) = C(\vec{k}, k^0) e^{-ik_\mu x^\mu} \quad \text{with} \quad k^0 = \pm\sqrt{\vec{k}^{\,2} + m^2},$$

so there are two solutions for each value of $\vec{k}$, one with $k^0 = \omega$ and one with $k^0 = -\omega$, with $\omega = +\sqrt{\vec{k}^{\,2} + m^2}$. So the general solution will be an integral $\int d^3k$ of solutions with arbitrary coefficients $C(\vec{k}, \omega) = \frac{1}{(2\pi)^3}\frac{1}{\sqrt{2\omega}} a(\vec{k})$, multiplying $e^{-i\omega t + i\vec{k}\cdot\vec{x}}$, together with negative energy modes which we will write in terms of $b(\vec{k}) := (2\pi)^3 \sqrt{2\omega} C(-\vec{k}, -\omega)$ for the coefficients multiplying $e^{i\omega t - i\vec{k}\cdot\vec{x}}$. So

$$\phi(x^\mu) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega}} \left[a(\vec{k}) e^{-ik\cdot x} + b(\vec{k}) e^{ik\cdot x} \right].$$

This is the correct form for the general scalar solution of the Klein-Gordon equation, but without the restriction that we have a single *real* field. To make $\phi(\vec{x}, t)$ real, we can insist on $b(\vec{k}) = a^*(\vec{k})$.

We may treat $a(\vec{k})$ as the dynamical degrees of freedom that need to be quantized, but first we need to say something about complex degrees of freedom. In the Lagrangian mechanics we have used, the assumption was made that the degrees of freedom are real variables. A complex variable could be considered as two real variables, so if $z = x + iy$ is a complex degree of freedom, we could treat $L(z, z^*, \dot{z}, \dot{z}^*)$ as $\tilde{L}(x, y, \dot{x}, \dot{y})$. We might wonder what it means to vary the Lagrangian with respect to z and z^* independently. By the chain rule,

$$\begin{aligned} \left.\frac{\partial \tilde{L}}{\partial x}\right|_y &= \left.\frac{\partial L}{\partial z}\right|_{z^*} + \left.\frac{\partial L}{\partial z^*}\right|_z \\ \left.\frac{\partial \tilde{L}}{\partial y}\right|_x &= i\left.\frac{\partial L}{\partial z}\right|_{z^*} - i\left.\frac{\partial L}{\partial z^*}\right|_z \end{aligned}$$

and similarly for $\dot{z}$ and $\dot{z}^*$. So if we naïvely assume we can define π and π^* by varying z and z^* independently, $\pi = \left.\frac{\partial L}{\partial \dot{z}}\right|_{z,z^*,\dot{z}^*} = \frac{1}{2}\left(\left.\frac{\partial \tilde{L}}{\partial \dot{x}}\right|_{\dot{y}} - i\left.\frac{\partial \tilde{L}}{\partial \dot{y}}\right|_{\dot{x}}\right) = \frac{1}{2}(\pi_x - i\pi_y)$, and $\frac{\partial L}{\partial z} = \frac{1}{2}\left(\left.\frac{\partial \tilde{L}}{\partial x}\right|_y - i\left.\frac{\partial \tilde{L}}{\partial y}\right|_x\right)$, So the Euler Lagrange equation from naïvely varying z and z^* independently,

$$\frac{d}{dt}\pi - \frac{\partial L}{\partial z} = \frac{1}{2}\left[\left(\frac{d}{dt}\pi_x - \frac{\partial L}{\partial x}\right) - i\left(\frac{d}{dt}\pi_y - \frac{\partial L}{\partial y}\right)\right],$$

which does indeed vanish from the two known equations from varying x and y independently. Similarly for the variation with respect to z^*. The Hamiltonian

$$H = \pi_x \dot{x} + \pi_y \dot{y} - L = \pi\dot{z} + \pi^*\dot{z}^* - \tilde{L}.$$

Now let us consider the real scalar field and treat $\phi(\vec{x})$ and its conjugate momentum

$$\pi(x) = \dot{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(-i\omega_k a(\vec{k}) e^{-ik\cdot x} + i\omega_k a^\dagger(\vec{k}) e^{ik\cdot x}\right)$$

as quantum mechanical fields which might not commute, which also means the coefficients $a(\vec{k})$ and $a^\dagger(\vec{k})$ may not commute. As the conjugate momentum to $\phi(\vec{x})$ is $\pi(\vec{x})$ quite independently of the degrees of freedom at other points $\vec{x}\,' \neq \vec{x}$, we expect $\phi(\vec{x})$ and $\pi(\vec{x}\,')$ to commute except when $\vec{x} = \vec{x}\,'$, and, as these are densities, we will need a Dirac delta function. Of course the $\phi(\vec{x})$ commute with each other, as they are the dynamical coordinates, and the $\pi(\vec{x})$ commute with each other, as they are the canonical momenta. Thus we assume quantization means

$$[\phi(\vec{x}),\phi(\vec{x}\,')]=0, \qquad [\phi(\vec{x}),\pi(\vec{x}\,')]=i\delta^3(\vec{x}-\vec{x}\,'), \qquad [\pi(\vec{x}),\pi(\vec{x}\,')]=0 \quad (2)$$

Notice these commutation relations are supposed to be taken at equal times, $t=t'$.

Expanding the fields in terms of a and $a^\dagger$, and doing (or undoing) the double Fourier transform, you will show for homework that

$$[a(\vec{k}),a(\vec{k}\,')]=0, \quad [a(\vec{k}),a^\dagger(\vec{k}\,')]=(2\pi)^3\delta^3(\vec{k}-\vec{k}\,'), \quad [a^\dagger(\vec{k}),a^\dagger(\vec{k}\,')]=0. \quad (3)$$

The Hamiltonian at $t=0$ is

$$H=\int d^3x\frac{1}{2}\left(\pi^2(\vec{x})+(\vec{\nabla}\phi)^2+m^2\phi^2\right).$$

Using the expansion of $\pi(\vec{x})$ and $\phi(\vec{x})$, and also

$$\vec{\nabla}\phi(x)=\int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(i\vec{k}a(\vec{k})e^{-ik\cdot x}-i\vec{k}a^\dagger(\vec{k})e^{ik\cdot x}\right),$$

we see that the Hamiltonian is

$$\begin{aligned}
H &= \frac{1}{2}\int d^3x\int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\int\frac{d^3k'}{(2\pi)^3\sqrt{2\omega_k'}} \\
&\qquad \Big[a(\vec{k})a(\vec{k}\,')e^{-i(\vec{k}+\vec{k}\,')\cdot\vec{x}}\left(-\omega_k\omega_k'-\vec{k}\cdot\vec{k}\,'+m^2\right) \\
&\qquad\quad +a(\vec{k})a^\dagger(\vec{k}\,')e^{-i(\vec{k}-\vec{k}\,')\cdot\vec{x}}\left(\omega_k\omega_k'+\vec{k}\cdot\vec{k}\,'+m^2\right) \\
&\qquad\quad +a^\dagger(\vec{k})a(\vec{k}\,')e^{i(\vec{k}-\vec{k}\,')\cdot\vec{x}}\left(\omega_k\omega_k'+\vec{k}\cdot\vec{k}\,'+m^2\right) \\
&\qquad\quad +a^\dagger(\vec{k})a^\dagger(\vec{k}\,')e^{i(\vec{k}+\vec{k}\,')\cdot\vec{x}}\left(-\omega_k\omega_k'-\vec{k}\cdot\vec{k}\,'+m^2\right)\Big] \\
&= \frac{1}{2}(2\pi)^3\int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\int\frac{d^3k'}{(2\pi)^3\sqrt{2\omega_k'}}\Big[\\
&\qquad a(\vec{k})a(\vec{k}\,')\delta^3(\vec{k}+\vec{k}\,')\left(-\omega_k\omega_k'-\vec{k}\cdot\vec{k}\,'+m^2\right) \\
&\qquad\quad +a(\vec{k})a^\dagger(\vec{k}\,')\delta^3(\vec{k}-\vec{k}\,')\left(\omega_k\omega_k'+\vec{k}\cdot\vec{k}\,'+m^2\right) \\
&\qquad\quad +a^\dagger(\vec{k})a(\vec{k}\,')\delta^3(\vec{k}-\vec{k}\,')\left(\omega_k\omega_k'+\vec{k}\cdot\vec{k}\,'+m^2\right)
\end{aligned}$$

$$\begin{aligned}
&\qquad\quad +a^\dagger(\vec{k})a^\dagger(\vec{k}\,')\delta^3(\vec{k}+\vec{k}\,')\left(-\omega_k\omega_k'-\vec{k}\cdot\vec{k}\,'+m^2\right)\Big] \\
&= \frac{1}{2}\int\frac{d^3k}{(2\pi)^3 2\omega_k}\Big[\\
&\qquad a(\vec{k})a(-\vec{k})\left(-\omega_k^2+\vec{k}^{\,2}+m^2\right) \\
&\qquad\quad +a(\vec{k})a^\dagger(\vec{k})\left(\omega_k^2+\vec{k}^{\,2}+m^2\right) \\
&\qquad\quad +a^\dagger(\vec{k})a(\vec{k})\left(\omega_k^2+\vec{k}^{\,2}+m^2\right) \\
&\qquad\quad +a^\dagger(\vec{k})a^\dagger(-\vec{k})\left(-\omega_k^2+\vec{k}^{\,2}+m^2\right)\Big] \\
&= \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\left[a(\vec{k})a^\dagger(\vec{k})+a^\dagger(\vec{k})a(\vec{k})\right]\omega_k \\
&= \int\frac{d^3k}{(2\pi)^3}\omega_k\left[a^\dagger(\vec{k})a(\vec{k})+\frac{1}{2}\delta^3(0)\right].
\end{aligned}$$

Notice that the Hamiltonian separates, each spatial momentum component $\vec{k}$ decoupling from the others and entering as a simple harmonic oscillator. However it is a bit disconcerting to have the zero point energy, $\frac{1}{2}\omega_{\vec{k}}$, for all of the infinite number of normal modes. As long as we avoid general relativity, and the coupling of this energy to gravitation, we can ignore this constant, though infinite, contribution to the energy of every state in the system — only energy differences have any effect. So we will drop this constant and write

$$H=\int\frac{d^3k}{(2\pi)^3}\,\omega_{\vec{k}}\,a^\dagger(\vec{k})\,a(\vec{k}).$$