Physics 613 Lecture 5 Feb. 6, 2014

1 P, C and T

We have discussed how ψ transforms under *proper*[1] *isochronous*[2] Lorentz transformations[3]. Now we turn our attention to three discrete transformations which might be symmetries: parity, time-reversal, and charge conjugation.

The transformation $\vec{x} \to \vec{x}\,' = -\vec{x}$, $t \to t' = t$ is called parity. It satisfies the general Lorentz transformation constraint but has determinant -1, so is not proper, though it is isochronous. Various classical physical properties have fixed behavior under parity, which is to say that we can define transformations of these quantities so as to maximize the possibility that our physics is invariant under parity. Three-dimensional scalars such as mass, charge, time and charge density are unchanged (though for fields, the argument changes). Polar vectors such as $\vec{x}$, velocity, $\vec{p}$, $\vec{E}$, force and acceleration change sign under parity, while pseudovectors such as angular momentum, spin $\vec{S}$ and magnetic field $\vec{B}$ do not. A quantity like helicity, $h = \vec{p} \cdot \vec{S}$ which is invariant under rotations but changes sign under parity is called a *pseudoscalar*. Cross products reverse sign under parity, which is why $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ requires that $\vec{E}$ is a vector but $\vec{B}$ is a pseudovector.

Pseudoscalars behave as scalars, and pseudovectors as vectors, under rotations.

We expect a field which behaves as a scalar under rotations to transform $\phi \to \phi'$ with $\phi'(\vec{x}\,') = \pm\phi(\vec{x})$ under parity. The phase is ± 1 because applying parity twice is the identity transformation and should leave ϕ unchanged. The field is scalar or pseudoscalar if the $\pm$ is $+1$ or -1 respectively. But a field with non-zero spin, which has more than one component, might have the components transform into each other. Let us consider a Dirac spinor, and see if we can define $\psi'_a(\vec{x}\,') = M_{ab}\psi_b(\vec{x})$ so that if $\psi(\vec{x})$ satisfies the Dirac equation $(i\gamma^\mu\partial_\mu - m)\,\psi(x) = 0$, the transformed one will also. So if we apply the Dirac operator, will it vanish?

$$0 \stackrel{?}{=} \left(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m\right)\psi'(x') = \left(i\gamma^\mu \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu - m\right) M\psi(x).$$

The $\partial x^\nu / \partial x'^\mu$ is zero unless $\mu = \nu$, and is $+1$ if they are zero but -1 if they are spatial (1,2,3). So if the matrix M satifies $[\gamma^0, M] = 0 = \{\gamma^k, M\}$, so that pulling M to the left through the γ^μ changes the signs of the spatial components but not the time component, we are left with $M\,(i\gamma^\mu\partial_\mu - m)\,\psi(x) = 0$, and our tranformed ψ' satisfies the same physics. Note $M = \gamma^0$ does exactly that, so we define $\psi'(x') = \gamma^0\psi(x)$.

We could have just as well introduced a phase η_P into M. Applying parity twice is the identity acting on x^μ, so it is convient to think $P^2\psi = \psi$, and we should have[4] $\eta_P = \pm 1$. We will choose $+1$.

Note if we had included an interaction with the electromagnetic field, the same sign change for $\partial_\mu \to \partial'_\mu$ also applies to $A_\mu \to A'_\mu$, as A_0 is unchanged but $\vec{A}'(x') = -\vec{A}_k(x)$, so $\vec{A}$ must be a polar vector, consistent with $\vec{B} = \vec{\nabla} \times \vec{A}$.

How do $\psi^\dagger$, $\bar\psi = \psi^\dagger\gamma^0$, and bilinears like $T^{\mu_1\cdots\mu_n} = \bar\psi\left(\prod_{j=1}^n \gamma^{\mu_j}\right)\psi$ transform? By taking the hermitean conjugate of $\psi'(x') = \gamma^0\psi(x)$ we have $\psi'^\dagger(x') = \psi^\dagger(x)\gamma^0$ as γ^0 is hermitean, and $\bar\psi'(x') = \bar\psi(x)\gamma^0$. For a bilinear

$$\bar\psi'(x')\left(\prod_{j=1}^n \gamma^{\mu_j}\right)\psi'(x') = \bar\psi(x)\left(\prod_{j=1}^n \left[\gamma^0\gamma^{\mu_j}\gamma^0\right]\right)\psi(x) = \pm\bar\psi(x)\left(\prod_{j=1}^n \gamma^{\mu_j}\right)\psi(x),$$

with a minus sign if there are an odd number of spatial indices μ_j and a plus sign otherwise. In other words, $T^{\mu_1\cdots\mu_n}$ transforms as one would expect for a tensor.

Can we not have a pseudotensor? If we define a new gamma matrix

$$\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$$

[1]Taking the determinant of $g_{\mu\nu} = \Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu g_{\rho\sigma}$ (Homework 1 Problem 2a) we see that $\det\Lambda^{\cdot}{}_{\cdot} = \pm 1$. Those with determinant $+1$ are called proper

[2]From the same equation, $g_{00} = 1 = \Lambda^\rho{}_0\Lambda^\sigma{}_0 g_{\rho\sigma} = (\Lambda^0{}_0)^2 - \sum_k(\Lambda^k{}_0)^2$ we see $(\Lambda^0{}_0)^2 \geq 1$. Those with $\Lambda^0{}_0 \geq 1$ are called isochronous, as they preserve the direction of time for timelike paths, while those with $\Lambda^0{}_0 \leq -1$ reverse the direction of time.

[3]Any Lorentz transformation reachable by continuous rotation and/or acceleration from the identity $\Lambda^\mu{}_\nu = \delta^\mu_\nu$ must have its determinant and its $\Lambda^0{}_0$ matrix element vary continuously, and therefore cannot jump the gap between $+1$ and -1, so this *connected set* of Lorentz transformations are the proper isochronous ones.

[4]This is not quite convincing, because ψ is not an observable, and we could say only bilinears like $\psi^\dagger(\prod_j \gamma^{\mu_j})\psi$ need to return to the same value under P^2.

we see that γ^5 anticommutes with each of the γ^μ, and therefore it commutes with the $[\gamma^\alpha, \gamma^\beta]$ which generate proper isochronous Lorentz transformations. Thus inserting a γ^5 in a bilinear will not change its transformation properties under these connected Lorentz transformations, but it does introduce an odd number of spatial index γ^μ's, so it reverses the behavior under parity, converting T into a pseudotensor.

In particular, $\bar\psi\psi$ is a scalar and $i\bar\psi\gamma_5\psi$ is a pseudoscalar, $\bar\psi\gamma^\mu\psi$ is a vector and $\bar\psi\gamma^\mu\gamma_5\psi$ is an axial or pseudo vector.

How does parity act on our positive energy state $u(\vec p, s)$ and our negative energy state $v(\vec p, s)$? We expect to get states with $\vec p\,' = -\vec p$ and s unchanged. Using our Dirac representation, for which $\gamma^0 = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}$,

$$\begin{aligned} \mathbf{P}u(\vec p, s) &= \mathbf{P}\sqrt{E+m}\begin{pmatrix} \phi^s \\ \frac{\vec\sigma\cdot\vec p}{E+m}\phi^s \end{pmatrix} = \sqrt{E+m}\,\gamma^0\begin{pmatrix} \phi^s \\ \frac{\vec\sigma\cdot\vec p}{E+m}\phi^s \end{pmatrix} \\ &= \sqrt{E+m}\begin{pmatrix} \phi^s \\ -\frac{\vec\sigma\cdot\vec p}{E+m}\phi^s \end{pmatrix} = \sqrt{E+m}\begin{pmatrix} \phi^s \\ \frac{\vec\sigma\cdot\vec p\,'}{E+m}\phi^s \end{pmatrix} = u(\vec p\,', s), \end{aligned}$$

but

$$\begin{aligned} \mathbf{P}v(\vec p, s) &= \mathbf{P}\sqrt{E+m}\begin{pmatrix} \frac{\vec\sigma\cdot\vec p}{E+m}\chi^s \\ \chi^s \end{pmatrix} = \sqrt{E+m}\,\gamma^0\begin{pmatrix} \frac{\vec\sigma\cdot\vec p}{E+m}\chi^s \\ \chi^s \end{pmatrix} \\ &= \sqrt{E+m}\begin{pmatrix} \frac{\vec\sigma\cdot\vec p}{E+m}\chi^s \\ -\chi^s \end{pmatrix} = \sqrt{E+m}\begin{pmatrix} -\frac{\vec\sigma\cdot\vec p\,'}{E+m}\chi^s \\ -\chi^s \end{pmatrix} = -v(\vec p\,', s), \end{aligned}$$

Note the negative energy solution has the opposite parity from the positive energy one.

1.1 Scalar particles

I skipped over scalar particles and the Klein-Gordon equation when discussing interaction with the electromagnetic fields via minimal substitution. Let's consider it now.

Using the *covariant derivative* $D_\mu := \partial_\mu + iqA_\mu$ in place of the ordinary derivative in the Klein-Gordon equation, we have

$$0 = \left(D_\mu D^\mu + m^2\right)\phi = \left(\Box + iq(\partial_\mu A^\mu) + 2iqA^\mu\partial_\mu - q^2 A_\mu A^\mu + m^2\right)\phi.$$

We note that under parity, the scalar potential A^0 should be unchanged and the vector potential should be a polar vector so that $\vec B = \vec\nabla\times\vec A$ is a

pseudovector. Thus A^0 doesn't change sign while $\vec A$ does, which is the same as for the partial derivative, so the interaction terms are invariant under parity.

In the Schrödinger equation, the probability density $\psi^*\psi$ is conserved with a probability current $\vec j = \frac{-i\hbar}{2m}\left(\psi^*\vec\nabla\psi - (\vec\nabla\psi^*)\psi\right)$. For the relativistic Klein-Gordon equation, this leads to the conserved probability current of the free scalar, $j^\mu = i\phi^*\partial^\mu\phi - i\left(\partial^\mu\phi*\right)\phi$. Note this is real, and it is conserved, $\partial_\mu j^\mu = i\partial_\mu\phi^*\partial^\mu\phi + i\phi^*\partial^2\phi - i\left(\partial^2\phi^*\right)\phi - i\partial_\mu\phi^*\partial^\mu\phi = -m^2\phi^*\phi + m^2\phi^*\phi = 0$). Now if $\phi(\vec x, t)$ is a solution of the Klein-Gordon equation, so is $\phi^*(\vec x, t)$, but we see that interchanging $\phi \leftrightarrow \phi^*$ changes the sign of the current and the density j^0. So j cannot be the probability current. If we multiply it by q, and call this the charge of the particle, we can consider it to be the electric charge and current densities. We see that interchanging ϕ and ϕ^* is equivalent to changing the sign of q. If we also do minimal substitution, we get

$$j^\mu(\phi, A^\mu) = iq\phi^* D^\mu\phi - iq\left(D^\mu\phi^*\right)\phi = j^\mu(\phi, 0) - 2q^2\phi^*\phi A^\mu. \tag{1}$$

Now interchanging ϕ and ϕ^* and also changing the sign of A^μ, without changing the parameter q, changes the sign of the current consistently. This is called *charge conjugation*. If ϕ is a positive energy state, we see that the negative energy ϕ^* corresponds to a particle of opposite charge.

From Dirac and those who reinterpreted holes in the negative energy sea as antiparticles, we see that these states with the electron's charge $-e$ and momentum p^μ (with $p^0 < 0$) should be reinterpreted as antiparticles with a positron's charge $+e$ and momentum $-p^\mu$ (and therefore with positive energy). This means not only the energy but also the momentum is reversed, so the wave function $e^{iEt-\vec p\cdot\vec x}$ is complex conjugated. So we can define our charge-conjugated solution $\phi_C(\vec x, t) = \eta_C\phi^*(\vec x, t)$ Our Klein-Gordon equation with interactions will be invariant if we also say that A^μ changes sign under charge conjugation, $A^\mu_C(\vec x, t) = -A^\mu(\vec x, t)$.

If we have a solution ψ of the Dirac equation $0 = \left(i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu - m\right)\psi$ with charge q in an external field A^μ, can we find a charge-conjugated Dirac field $\psi_C(\vec x, t)$ which satisfies the Dirac equation with charge $-q$? We suspect something like $\psi_C(\vec x, t) = M\psi^*(\vec x, t)$ should work. We would like

$$0 = \left[i\gamma^\mu\partial_\mu - (-q)\gamma^\mu A_\mu - m\right]\psi_C(\vec x, t).$$

If we take the complex conjugate, we can ask if

$$\begin{aligned} 0 &\stackrel{?}{=} \left(-i\gamma^{\mu\,*}\partial_\mu + q\gamma^{\mu\,*}A_\mu - m\right)\psi^*_C(\vec x, t) \\ &= \left(-i\gamma^{\mu\,*}\partial_\mu + q\gamma^{\mu\,*}A_\mu - m\right)M^*\psi(\vec x, t). \end{aligned}$$

Note that, in our representation, all the γ matrices are real except γ^2, which is pure imaginary, so $-\gamma^{\mu\,*}\gamma^2 = \gamma^2\gamma^\mu$, so if $M^* = i\gamma^2 = M$, we have

$$M\left(i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu - m\right)\psi(\vec{x},t) = 0,$$

and ψ_C obeys the $-q$ Dirac equation, even with E&M interactions.

If we look at the transform of the $A = 0$ positive energy solution u, we get

$$\begin{aligned} u_C(\vec{p},s) &= \sqrt{E+m}\, i\gamma^2 \begin{pmatrix} \phi^{s*} \\ \vec{\sigma}^* \cdot \vec{p}\,\phi^{s*}/(E+m) \end{pmatrix} \\ &= \sqrt{E+m} \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} \phi^{s*} \\ \vec{\sigma}^* \cdot \vec{p}\,\phi^{s*}/(E+m) \end{pmatrix} \\ &= \sqrt{E+m} \begin{pmatrix} \vec{\sigma}\cdot\vec{p}(-i\sigma_2\phi^{s*})/(E+m) \\ -i\sigma_2\phi^{s*} \end{pmatrix} \\ &= \sqrt{E+m} \begin{pmatrix} \vec{\sigma}\cdot\vec{p}\,\chi^s/(E+m) \\ \chi^s \end{pmatrix} = v(\vec{p},s) \end{aligned}$$

with $\chi^s = -i\sigma_2\phi^{s*}$.

Similarly, the charge conjugate of $v(\vec{p},s)$ is $u(\vec{p},s)$.

In the Dirac picture, with negative energy states identified as the absence of particles going backwards in time as antiparticles, it is a bit hard to distinguish charge conjugation from time reversal. Classically, however time reversal changes the sign of $\vec{v}$, $\vec{p}$, and $\vec{j}$, but not accelerations or forces, and thus not $\vec{E}$, but it does change the sign of $\vec{B}$, partly because it is generated by currents. So $A^0 \to +A^0$ but $\vec{A} \to \vec{A}$. Going backwards in time we expect $\psi_T(\vec{x},t')$ is related to $\psi(\vec{x},-t)$ with the opposite momentum. This requires the complex conjugate, so we expect $\psi_T(\vec{x},t') = U_T\psi^*(t)$ The Dirac equation requires

$$0 = \left(i\gamma^\mu\partial'_\mu - q\gamma^\mu A'_\mu - m\right)\psi_T(t')$$

has a complex conjugate

$$\left(-i\gamma^{\mu*}\partial'_\mu - q\gamma^{\mu*}A'_\mu - m\right)U_T^*\psi(t) = 0.$$

As the complex conjugate on the γ's gives a minus sign for $\mu = 2$ only, and as the partial derivative gives one for $\mu = 0$, the first term changed sign for $\mu = 1$ and 3. As A^μ changes sign for spatial μ, the same is true for the second term. So if U_T^* anticommutes with γ^1 and γ^3 and commutes with the

others, we have $U_T^*\left(i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu - m\right)\psi(t)$ which is indeed zero. So we take $U_T = -i\gamma^1\gamma^3 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$.

We will discuss P, C, and T again after we reformulate our thinking along the lines of quantum field theory, where antiparticles will be on the same footing as particles and the confusion between C and T is cleared up.

Just a few words about applications now, however. As we have already seen, parity conservation in $\pi^0 \to \gamma + \gamma$ decay requires the pion to have negative intrinsic partity, and this effects the parity of 3 pion final states, as we discussed in τ decay. That the $\tau \to 3\pi$ and the $\theta \to 2\pi$ are the same particle showed that parity is not conserved by the weak interactions which enable these decays, but parity *is conserved* by the strong and electromagnetic interactions. The strong and electromagnetic interactions also conserve charge conjugation and time reversal invariance, but again the weak interactions do not. The weak interactions seem to violate P to the maximum extent possible, coupling only to left handed helicity states and not right handed states. They do come close to symmetry under the combined CP, violating it 1/1000 times as strongly as they violate P, so that this violation has been seen experimentally only in the very small mixing of the K^0 with its antiparticle and of the $B^0 = (d\bar{b})$ with its antiparticle, producing even and odd combinations with very slightly different masses.

As we shall see when we repeat this discussion more carefully in the context of quantum field theory, any Lorentz invariant QFT with local fields and a Hermitean hamiltonian and obeying the spin-statistics theorem must preserve the combined PCT symmetry.