Physics 613 Lecture 4 Feb. 4, 2014

1 More on Dirac Solutions

We have seen that the Dirac equation has solutions $\psi = \omega e^{-ip\cdot x}$, two with positive $p^0 = E = \sqrt{\vec{p}^2 + m^2}$ and two with negative $p^0 = -E$, with

$$\omega = \begin{pmatrix} \phi \\ \dfrac{\vec\sigma\cdot\vec p}{E+m}\,\phi \end{pmatrix} \text{ for } p^0 > 0, \qquad \text{and} \quad \omega = \begin{pmatrix} \dfrac{-\vec\sigma\cdot\vec p}{E+m}\chi \\ \chi \end{pmatrix} \text{ for } p^0 < 0.$$

The ϕ and χ are two component objects, so there are two independent solutions for each p^0. If $\phi \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we have spin in the positive z direction, which we might call $\phi_\uparrow$, and $\phi \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $\phi_\downarrow$. But it is often more useful, for particles not at rest, to quantize spin in the direction of $\vec p$, diagonalizing the helicity operator

$$h(\vec p) = \begin{pmatrix} \dfrac{\vec\sigma\cdot\vec p}{|\vec p|} & 0 \\ 0 & \dfrac{\vec\sigma\cdot\vec p}{|\vec p|} \end{pmatrix}$$

which has eigenvectors for $\phi = \phi_\pm$ if $\dfrac{\vec\sigma\cdot\vec p}{|\vec p|}\phi_\pm = \pm\phi_\pm$. These are helicity + and − respectively.

The normalization we usually use in non-relativistic quantum mechanics, with $\int d^3x\, \psi^\dagger\psi = 1$, is not convenient relativistically, as the integral over space at a fixed time is not a relativistic invariant. Instead, it will be useful to normalize our wave functions to $\omega^\dagger\omega = 2E$, and define

$$u(\vec p, s) = \omega(E, p, s) = \sqrt{E+m}\begin{pmatrix} \phi^s \\ \dfrac{\vec\sigma\cdot\vec p}{E+m}\phi^s \end{pmatrix},$$

$$v(\vec p, s) := \omega(-E, -\vec p, s) = \sqrt{E+m}\begin{pmatrix} \dfrac{\vec\sigma\cdot\vec p}{E+m}\chi^s \\ \chi^s \end{pmatrix},$$

for $s = 1, 2$, where $\phi^1 = \chi^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\phi^2 = \chi^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The reason for reversing the indices on χ and of defining v in terms of $-E$ $-\vec p$ is related to Dirac's understanding of the negative energy solutions.

What does it mean that there are negative energy solutions to the free Dirac equation? If there were never any interactions, perhaps we could just ignore them, and claim that physical states are the positive energy solutions. But if interactions are added, there will be transitions between free particle eigenstates, and particles can fall into or pop out of the negative energy states. In order to have positive energy solutions be stable in the presence of interactions, Dirac proposed that the sea of negative energy states is full in the vacuum state. Still, interactions might kick a negative energy solution into a positive energy one, leaving a hole in the vacuum which would have a positive charge (as we had removed one negative charge from the vacuum), giving a state with one extra electron and one positron. This is quite a nice prediction, but it does mean that we are no longer dealing with a Hilbert space with a fixed number of particles.

The idea that new stuff can be created in addition to the particles we originally had is not new. Consider what happens if two charged particles in empty space come close to each other and scatter. If we ignored radiation and considered only the Schrödinger equation with the Coulomb potential, we could treat this as ordinary quantum mechanics of a two particle system. But we know that accelerating particles produce electromagnetic radiation, so in fact the final state will consist not only of the two deflected particles but also an electromagnetic field. That field can be considered quantum mechanically, either as a quantum theory of the electromagnetic field, or as having an undetermined number of photons. In fact, the two things are the same, according to quantum field theory.

As this idea first presents itself for the electromagnetic field, let us first review what we know about it classically. Maxwell and Lorentz[1] gave us the correct relativistic theory well before relativity was proposed. Maxwell tells us[2]

$$\vec\nabla\cdot\vec B = 0, \qquad \text{No magnetic charge}$$

[1] Actually, the magnetic part of the Lorentz force was correctly described by Heaviside in 1889 and possibly by Maxwell in 1865, well before Lorentz combined it with the Coulomb electric part in 1892.

[2] In rationalized MKS Heaviside-Lorentz units, using $c = 1$.

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= \rho, && \text{Gauss' law}\\
\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t} &= 0, && \text{Faraday's law}\\
\vec{\nabla}\times\vec{B}-\frac{\partial\vec{E}}{\partial t} &= \vec{j} && \text{Ampère/Maxwell.}
\end{aligned}$$

where ρ is the charge density and $\vec{J}$ the current density. The first two of these equations do not look like equations of motion, as they have no time derivatives, but are constraint equations. The first tells us[3] that there is a vector field $\vec{A}(\vec{x})$ such that $\vec{B}=\vec{\nabla}\times\vec{A}$. Then writing Faraday in terms of $\vec{A}$, $\vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0$, which again says there is a scalar field $-V$ whose gradiant is the piece in parentheses, or

$$\vec{E}=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}.$$

This reduces the 6 fields $\vec{E}$ and $\vec{B}$ to the four fields $A^\mu=\left(V,\vec{A}\right)$. With $\vec{E}$ and $\vec{B}$ defined in terms of A^μ, the two sourceless laws of Maxwell are automatically satisfied. Let us define the field-strength tensor

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu.$$

This is antisymmetric in $\mu\leftrightarrow\nu$, so the diagonal elements are zero. If one of the indices is 0, we have

$$F^{0j}=\frac{\partial A^j}{\partial t}+\partial_j V=-E_j=-F^{j0},$$

and if both are spacelike,

$$F^{jk}=-\partial_j A^k+\partial_k A^j=-\epsilon_{jk\ell}B^\ell.$$

Then

$$\begin{aligned}
\partial_\mu F^{\mu j} &= -\frac{\partial E_j}{\partial t}+(\vec{\nabla}\times\vec{B})_j=\vec{j}\\
\partial_\mu F^{\mu 0} &= \vec{\nabla}\cdot\vec{E}=\rho.
\end{aligned}$$

[3]As Minkowski space is contractable, and so is space for a given time, the Poincaré Lemma tells us that the a divergence-free vector field on space (a closed two-form) is exact, that is, the curl of a vector field $\vec{A}(\vec{x})$ (a one-form).

Naturally we define the 4-current $J^\mu=\left(\rho,\vec{j}\right)$ and find

$$\partial_\mu F^{\mu\nu}=J^\nu.$$

This has a wonderful consequence:

$$\partial_\nu J^\nu=\underbrace{\partial_\nu\partial_\mu}_{\text{symmetric}}\underbrace{F^{\mu\nu}}_{\text{antisymmetric}}=0,$$

showing that the electric current is automatically conserved and hence so is charge.

As we will see later, understanding quantum field theory begins with the Lagrangian form of the mechanics, so we may ask what Lagrangian generates Maxwell's equation. In general, if the lagrangian density $\mathcal{L}$ depends on fields η_j, with $\eta_{j,\mu}:=\partial_\mu\eta_j$ the equations of motion are

$$\frac{d}{dx^\mu}\frac{\partial\mathcal{L}}{\partial\eta_{j,\mu}}-\frac{\partial\mathcal{L}}{\partial\eta_j}=0.$$

For electromagnetism, the dynamical fields are A_μ, and the lagrangian density is

$$\mathcal{L}=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-A_\nu J^\nu.$$

The first term depends only on $A_{\mu,\nu}$ and not on A_μ itself, so using

$$\frac{\partial F_{\mu\nu}}{\partial A_{\rho,\sigma}}=\frac{\partial F_{\mu\nu}}{\partial(\partial_\sigma A_\rho)}=\delta^\sigma_\mu\delta^\rho_\nu-\delta^\sigma_\nu\delta^\rho_\mu \quad\text{we see}\quad \frac{\partial\mathcal{L}}{\partial A_{\rho,\sigma}}=F^{\rho\sigma}\quad\text{so}\quad \partial_\sigma F^{\sigma\rho}=J^\rho,$$

in agreement with what we want.

The replacement by the physically directly observable fields $\vec{E}$ and $\vec{B}$ by the four dimensional vector potential A^μ simplifies the dynamics some, but it introduces another strangeness. If we consider what happens if we add to $A_\mu(x^\rho)$ the 4-gradient of a scalar function[4] $\Lambda(x^\rho)$,

$$A_\mu(x^\rho)\longrightarrow A_\mu(x^\rho)+\partial_\mu\Lambda(x^\rho), \tag{1}$$

the electric and magnetic fields

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\longrightarrow\partial_\mu A_\nu+\partial_\mu\partial_\nu\Lambda-\partial_\nu A_\mu-\partial_\nu\partial_\mu\Lambda=\partial_\mu A_\nu-\partial_\nu A_\mu=F_{\mu\nu}$$

[4]Note this Λ has nothing to do with our Lorentz transformations, also called Λ.

so they are unchanged. Even if we look at the action, which is the four dimensional integral of the lagrangian density $\mathcal{L}$, the change takes

$$-A_\nu J^\nu \longrightarrow -A_\nu J^\nu - (\partial_\nu \Lambda)\, J^\nu$$

so the change in the action is

$$\delta S = -\int d^4x\ (\partial_\nu\Lambda)\, J^\nu = \int d^4x\, \Lambda\, (\partial_\nu J^\nu) - \int_\infty (\Lambda J^\mu)\, d\Sigma_\mu,$$

where we have integrated by parts, and the $\int_\infty$ is an integral at the boundary of the space-time volume we have considered, presumably all infinitely distant[5] and where we can assume our fields have extenuated away. Then if our source J^ν satisfies the conservation requirement $\partial_\nu J^\nu$, we see that the change in A_ν has no physical effect.

The transformation (1) is a *local gauge transformation*. Notice that its variation is independent at each point in spacetime, unlike a global transformation like a rotation, Lorentz transformation or isospin transformation.

1.0.1 Interaction with matter

In classical nonrelativistic mechanics, a particle of charge q has a Hamiltonian

$$H = \frac{(\vec{p} - q\vec{A}(\vec{x}))^2}{2m} + qV(\vec{x}).$$

The hamilton equations of motion are then

$$\begin{aligned} \frac{dx_j}{dt} &= \frac{\partial H}{\partial p_j} = \frac{1}{m}(p_j - qA_j) \\ \frac{dp_j}{dt} &= -\frac{\partial H}{\partial x_j} = -q\partial_j V + \frac{q}{m}\sum_k (p_k - qA_k)\partial_j A_k = -q\partial_j V + q\sum_k \frac{dx_k}{dt}\partial_j A_k \end{aligned}$$

where I have used the first equation, $\vec{p} = m\dot{\vec{x}} + q\vec{A}$ in the second, which then gives

$$m\frac{d^2}{dt^2}x_j + q\frac{dA_j(\vec{x})}{dt} = -q\partial_j V + q\sum_k \frac{dx_k}{dt}\partial_j A_k,$$

[5] Another reason to ignore this hypersurface integral is that in deriving equations of motion classically, or doing a functional integral quantum-mechanically, we are told to keep the dynamical quantities at the surface constant and only vary physics in the interior of the region.

The second term is a total derivative, so is $q\frac{\partial\vec{A}}{\partial t} + q(\dot{\vec{x}}\cdot\vec{\nabla})\vec{A}$. Bringing that to the right, we have

$$m\frac{d^2}{dt^2}x_j = -q\partial_j V + q\sum_k \frac{dx_k}{dt}\partial_j A_k - q\frac{\partial A_j}{\partial t} - q\sum_k \frac{dx_k}{dt}\partial_k A_j$$

With $\vec{E} = -\vec{\nabla}V - \frac{d\vec{A}}{dt}$ and $\epsilon_{jk\ell}B_\ell = \epsilon_{jk\ell}\epsilon_{\ell mn}\partial_m A_n = \partial_j A_k - \partial_k A_j$, we see that

$$m\frac{d^2}{dt^2}x_j = q\left(E_j + \epsilon_{jk\ell}\frac{dx_k}{dt}B_\ell\right),$$

or

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right).$$

Noting that $V = A^0$ we see that we have really used just the free particle non-relativistic equation $p^0 = \vec{p}^2/2m$ and substituted $p^\mu \longrightarrow p^\mu - qA^\mu$, which is known as em minimal substitution or *minimal coupling*. As a quantum mechanical statement with $p_\mu \to i\hbar\partial_\mu$, when we apply this to the Dirac equation we have

$$(i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu - m)\psi = 0. \tag{2}$$

One very interesting consequence comes from examining the effect of a uniform magnetic field $\vec{B} = B\hat{e}_z$ expressed in terms of the vector potential $\vec{A} = \frac{1}{2}B(-y, x, 0)$ or $A^j = -\frac{B}{2}\epsilon_{jk3}r_k$. In ordinary quantum mechanics,

$$H = \frac{p^2}{2m} \to \frac{(\vec{p} - q\vec{A})^2}{2m} + qA^0$$

$$E\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{i\hbar q}{2m}\left(\vec{\nabla}\cdot\vec{A} + 2\vec{A}\cdot\vec{\nabla}\right) + \frac{q^2}{m^2}\vec{A}^2 + qA^0\right)\psi.$$

Noting that $\vec{\nabla}\cdot\vec{A} = 0$ and $2\hbar\vec{A}\cdot\vec{\nabla} = -\hbar B\epsilon_{jk3}r_k\partial_j = i\vec{B}\cdot\hat{\vec{L}}$, we see that the magnetic term is $-\frac{q}{2m}\vec{B}\cdot\vec{L}$. Comparing to the usual expression for the energy due to a magnetic moment, $E = -\vec{\mu}\cdot\vec{B}$, we see that orbital angular momentum contributes a magnetic moment

$$\vec{\mu} = \frac{q}{2m}\vec{L}.$$

The quantity $\frac{\hbar q}{2m}$ for an electron is called the *Bohr magneton* ($= 9.27\times 10^{-24}$ J/T).

This discussion did not consider intrinsic spin, but if we thought the same relation should hold for it, with $\vec{L} \to \hbar\vec{S} = \frac{\hbar}{2}\vec{\sigma} \sim \pm\frac{\hbar}{2}$. This turns out to be approximately half what we find measuring the energy difference in the ground state of hydrogen. The *gyromagnetic ratio* fudge factor g was introduced, $\vec{\mu} = g\frac{\hbar q}{2m}\vec{S}$, and experiment found g to be very close to 2.

Let us look at the Dirac equation with minimal substitution. Premultiply the Dirac equation (2) by $(i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu + m)$ to get

$$\left[(i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu)^2 - m^2\right]\psi = 0.$$

Again using $A^\mu = \frac{1}{2}B(0,-y,x,0)$, we have

$$\left(-\partial_\mu\partial^\mu - iq\gamma^\mu\gamma^\nu\left(\partial_\mu A_\nu\right) - iq\left\{\gamma^\mu,\gamma^\nu\right\}A_\nu\partial_\mu + q^2A_\mu A^\mu - m^2\right)\psi = 0.$$

As $\partial_k A_j = \frac{1}{2}B\epsilon_{jk3}$ and the zero'th components vanish, and as $\gamma^k\gamma^j\epsilon_{jk3} = 2i\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix}$, the second term is $iq\vec{\sigma}\cdot\vec{B}$. The third term is $-2iqA^\mu\partial_\mu = 2qA^\mu p_\mu = -2q\vec{A}\cdot\vec{p}$. This is necessary to change $\vec{p}^2$ into $m^2(\dot{\vec{x}})^2$, and is not part of the interaction with $\vec{B}$. The fourth term is negligible[6], so we have $(p^2 + iq\vec{\sigma}\cdot\vec{B})\psi = 0$. Then

$$E\psi = \sqrt{\vec{p}^2 + m^2 - iq\vec{\sigma}\cdot\vec{B}}\,\psi \sim \left(\sqrt{\vec{p}^2+m^2} - \frac{iq}{2m}\vec{\sigma}\cdot\vec{B}\right)\psi.$$

Thus the extra energy from the magnetic field is $E = -\frac{iq}{2m}\vec{\sigma}\cdot\vec{B}$, twice the result from non-relativistic considerations. The result $g = 2$ which comes unexpectedly from the Dirac equation was a great triumph, establishing its correctness. As we shall see, the corrections to this from quantum field theory (g = 2.002319304364) is the greatest triumph of QED.

[6] If $B \sim 1$ T and we are interested in a region of size ~ 1 Å, for an electron $qA^\mu/m_e \sim 10^{-7}$