Physics 613 Homework #6
Due March 10, 2014 at 4:00 EST

1: The positive and negative energy solutions of the free Dirac equation were defined to be

$$u(\vec{p},r) = \sqrt{E+m}\begin{pmatrix} \phi^r \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\phi^r \end{pmatrix}, \qquad v(\vec{p},r) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^r \\ \chi^r \end{pmatrix},$$

with $\phi^{r\,\dagger}\phi^s = \chi^{r\,\dagger}\chi^s = \delta_{rs}$, and $E = +\sqrt{\vec{p}^2+m^2}$.

Show:

(a) $u^\dagger(\vec{p},r)u(\vec{p},s) = 2E\delta_{rs}$, (b) $v^\dagger(\vec{p},r)v(\vec{p},s) = 2E\delta_{rs}$,

(c) $\bar{u}(\vec{p},r)u(\vec{p},s) = 2m\delta_{rs}$, (d) $\bar{v}(\vec{p},r)v(\vec{p},s) = -2m\delta_{rs}$,

(e) $\bar{u}(\vec{p},r)v(\vec{p},s) = 0$ (f) $\bar{v}(\vec{p},r)u(\vec{p},s) = 0$

(g) $u^\dagger(\vec{p},r)v(\vec{p},s) \neq 0$ (h) $v^\dagger(\vec{p},r)u(\vec{p},s) \neq 0$

(j) $u^\dagger(\vec{p},r)v(-\vec{p},s) = 0$ (k) $v^\dagger(\vec{p},r)u(-\vec{p},s) = 0$

2: We previously considered contracting $u^\dagger u$ and the like to form scalars. But we can also take outer products, $uu^\dagger$, which is not a scalar but a tensor. The positive and negative energy solutions of the free Dirac equation were defined to be

$$u(\vec{p},r) = \sqrt{E+m}\begin{pmatrix} \phi^r \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\phi^r \end{pmatrix}, \qquad v(\vec{p},r) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^r \\ \chi^r \end{pmatrix},$$

with $\phi^{r\,\dagger}\phi^s = \chi^{r\,\dagger}\chi^s = \delta_{rs}$, $\sum_s \phi^s\phi^{s\,\dagger} = \mathbb{1}$, and $E = +\sqrt{\vec{p}^2+m^2}$.

Show

(a) $\sum_{s=1}^2 u(\vec{p},s)\bar{u}(\vec{p},s) = \gamma^\mu p_\mu + m$.

(b) $\sum_{s=1}^2 v(\vec{p},s)\bar{v}(\vec{p},s) = \gamma^\mu p_\mu - m$.

3: Using the anticommutation relations of the Dirac creation and annihilation operators

$$\left\{\hat{c}_r(\vec{k}), \hat{c}_s(\vec{q})\right\} = 0, \quad \left\{\hat{c}_r(\vec{k}), \hat{c}_s^\dagger(\vec{q})\right\} = (2\pi)^3\delta_{rs}\delta^3(\vec{k}-\vec{q}), \quad \left\{\hat{c}_r^\dagger(\vec{k}), \hat{c}_s^\dagger(\vec{q})\right\} = 0.$$

$$\left\{\hat{d}_r(\vec{k}), \hat{d}_s(\vec{q})\right\} = 0, \quad \left\{\hat{d}_r(\vec{k}), \hat{d}^\dagger_s(\vec{q})\right\} = (2\pi)^3 \delta_{rs} \delta^3(\vec{k}-\vec{q}), \quad \left\{\hat{d}^\dagger_r(\vec{k}), \hat{d}^\dagger_s(\vec{q})\right\} = 0.$$

and the expression for the Dirac field,

$$\hat{\psi}_a(\vec{x}) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \sum_{s=1}^{2} \left[\hat{c}_s(\vec{k})\, u_a(\vec{k}, s)\, e^{-ik_\mu x^\mu} + \hat{d}^\dagger_s(\vec{k})\, v_a(\vec{k}, s)\, e^{ik_\mu x^\mu}\right],$$

with $k_0 = \omega_k = \sqrt{\vec{k}^{\,2} + m^2}$, find the anticommutation relations at equal time between $\hat{\psi}_a(\vec{x}, t)$ and $\hat{\psi}^\dagger_b(\vec{y}, t)$.