Physics 613 Spring, 2014
Homework #4. Due Feb. 24 at 4:00

1: In non-relativistic physics, we normalize our states in terms of the integration over d^3x, or in momentum space, we have $\int d^3k \left|\vec{k}\right\rangle \left\langle\vec{k}\right| = 1$. But d^3k is not a Lorentz invariant integration measure. Show

(a) d^4k is an invariant measure for proper Lorentz transformations. That is, if $k'^\mu = \Lambda^\mu{}_\nu k^\nu$ for a proper Lorentz transform Λ, $dk'^0 dk'^1 dk'^2 dk'^3 = dk^0 dk^1 dk^2 dk^3$.

(b) As $\delta(k^2 - m^2)$ is Lorentz invariant, that means $\int d^4k\, \delta(k^2 - m^2)$ is Lorentz invariant. With $m^2 \geq 0$, when $\delta(k^2 - m^2)$ is coupled with[1] $\Theta(k^0)$, the combination is also invariant under isochronous Lorentz transformations. Thus we have $d^4k\Theta(k^0)\delta(k^2 - m^2)$ is invariant under proper isochronous Lorentz transformations. Show this means

$$\int \frac{d^3k}{(2\pi)^3 2\omega_k} \quad \text{is invariant}$$

for proper isochronous Lorentz transformations.

2: Evaluate for a free real scalar field the vacuum matrix element

$$\langle 0|\, \phi(\vec{x}\,', t')\phi(\vec{x}, t)\, |0\rangle\,.$$

You should reduce this to a single interval, and for spacelike separation, find the explicit answer in terms of the modified Bessel function $K_1\left(m\sqrt{-(\Delta x)^2}\right)$ Hint: Your answer should be invariant under space-time translation, thus a function only of $\vec{x}\,' - \vec{x}$ and $t' - t$, and it should be invariant under proper isochronous Lorentz transformations. You should be able to extend the $|k|$ integral to the whole real axis, and distort the contour around the cut. Then it may help to know

$$\int_1^\infty dv\, \frac{v}{\sqrt{v^2 - 1}}\, e^{-bv} = K_1(b).$$

[1] The Heaviside step function $\Theta(x) = 1$ for $x > 0$ and $= 0$ for $x < 0$. If necessary $\Theta(0) = 1/2$, but we never really need that.