Physics 613 Homework #10
Due April 21, 2014 at 4:00 EST

1: This problem concerns issues in Lie groups and their generators, which form a Lie algebra.

1. Consider elements A, B, and C of an associative[1] algebra[2]. Define the commutator as usual, $[A, B] := AB - BA$. Prove the Jacobi identity:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

2. The Lie algebra may, as a vector space, be given a basis T_j, with $j = 1, \dots n$, where n is the dimension of the Lie algebra, and also of the Lie group it generates. The Lie algebra is closed under commutation. The structure of the algebra is given by these structure constants[3] ${f_{jk}}^\ell$ defined by

$$[T_j, T_k] = \sum_\ell i {f_{jk}}^\ell T_\ell.$$

Find the implication for the structure constants that comes from the Jacobi identity.

3. The way a symmetry group acts on the states in physics is generally linearly, with each subspace of the states transforming within itself, the group transformations acting as matrices. Thus a d dimensional representation of a group is a mapping from the group G into $d \times d$ matrices, $g \in G \mapsto M(g) \in \mathbb{C}^{d \times d}$, with the property that if $g_3 = g_1 \circ g_2$, (the composition of applying g_2 and then applying g_1), then $M(g_3) = M(g_1)M(g_2)$.

[1] That is, $(AB)C = A(BC)$.

[2] An algebra (over a field, for us, always either the reals or the complex numbers) is a vector space with a bilinear product. Except for our Lie algebras, we will assume algebra implies it is associative, but not necessarily commutative.

[3] I am making a perhaps unnecessary improvement by writing ${f_{jk}}^\ell$ where the book uses $f_{jk\ell}$. Once one has chosen the basis of the Lie algebra in accord with part 6, there is no distinction.

In a Lie group the group elements can be written as exponentials of generators, $g(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{T}}$, where α is a real n dimensional vector. Then the representation M of the group induces a representation of the algebra,

$$M(T_j) = \lim_{\vec{\alpha}\to 0} \frac{\partial M(\vec{\alpha})}{\partial \alpha_j}.$$

Show that $[M(T_j), M(T_k)] = i{f_{jk}}^\ell M(T_\ell)$.

4. Show that $\left(M_{\text{adj}}(T_\ell)\right)_j{}^k = i{f_{j\ell}}^k$ is a representation[4] of the Lie algebra. It is called the *adjoint* representation.

5. For SU(N), the *fundamental* representation is just the unitary $N \times N$ matrices of determinant 1 that define the group. That this is a representation is trivial. The representation of the Lie algebra is then traceless hermitian matrices. For SU(3), the canonical basis is given in appendix M.4.5, and the corresponding structure constants are given there as well.

Every Lie algebra has a bilinear *Killing form*[5] which maps a pair of vectors in the Lie algebra into a real number. It is determined by its value on the basis vectors, of course, so we define

$$\beta_{jk} = \beta(T_j, T_k) = -\sum_{ab} {f_{aj}}^b {f_{bk}}^a.$$

For any representation, there is a similar

$$\textbf{trace form}: \qquad F(T_j, T_k) = \mathrm{Tr}(M(T_j)M(T_k)).$$

Show the Killing form is the trace form for the adjoint representation. Evaluate the Killing form and the trace form for the fundamental representation for SU(2) with $T_j = \frac{1}{2}\sigma_j$, where $\vec{\sigma}$ are the Pauli matrices.

6. So far we have said only that the T_j are basis vectors for the Lie algebra, so they are somewhat arbitrary. For any nonsingular real matrix $\mathbf{M}$,

[4] Note I am writing the matrix elements of a matrix as ${M_a}^b$ to be consistent with the format of contra- and co- variant indices, as will be clearer in part 6.

[5] Named after Wilhelm Killing, though if it kills vectors in the Lie algebra, it proves the algebra is not semi-simple. Whatever.

we could define an alternate set of generators $T'_j = \mathbf{M}_j{}^k T_k$. This will require a new set of structure constants f' and a new Killing form β'_{jk}. Find them. Then show that if the Killing form is a positive definite matrix, such a change of basis allows us to make it diagonal. The convention is to choose the basis vectors so that $\beta(T_j, T_k) = 2\delta_{jk}$.

7. Show that if that is done, the new structure constants, (dropping the primes) are antisymmetric in the last two indices, $f_{jk}{}^{\ell} = -f_{j\ell}{}^{k}$, which, together with the antisymmetry in the first two which is apparent from their definition, means $f_{jk}{}^{\ell}$ is totally antisymmetric in its three indices.

8. Show that with this choice of basis (with $\beta_{jk} = 2\delta_{jk}$),

$$C := \sum_{jk} \beta(T_j, T_k) T_j T_k$$

   is a Casimir operator. That is, show C commutes with any element of the Lie algebra. As a consequence, C takes on the same value for any vector within a given irreducible representation.