Physics 613 Spring, 2014
Homework #1. Due Feb. 3 at 4:00

1 Do Problem 1.3 in the 4th Ed. of Aitchison and Hay (Prob 2.2 in the 3rd Ed.) For part (d), estimate the He nucleus to have a radius of 1.5 fm.

2 Under a Lorentz transformation, any contravariant vector V^μ transforms linearly with a matrix $\Lambda^\mu{}_\nu$, in just the same way as the space-time coordinate x^μ.

$$V^\mu \to V'^\mu = \Lambda^\mu{}_\nu V^\nu$$

(summed on ν, of course). The requirement on Λ for this to be a Lorentz transformation is that the Minkowski dot product of two vectors,

$$V \cdot W = V^0 W^0 - \sum_{j=1}^{3} V^j W^j = V^\mu W_\mu$$

is invariant, that is, $V \cdot W = V' \cdot W'$.

(a) From the requirement that this hold for any two vectors, show that

$$g_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu g_{\rho\sigma}.$$

(b) Any finite Lorentz transformation in the connected part of the Lorentz group, that is, with $\det \Lambda^\mu{}_\nu = 1$ and $\Lambda^0{}_0 > 0$, can be written in terms of an infinitesimal generator $L^\mu{}_\nu$ and a finite parameter η. The 4×4 matrix $\Lambda^\mu{}_\nu = (\exp -i\eta L)^\mu{}_\nu$, where the exponential of a matrix is understood to mean the power series with matrix multiplication. Each matrix factor in a product needs to have a first contravariant index and a second covariant index.

Show that the condition on the matrix L is thus

$$g_{\alpha\mu} L^\mu{}_\beta = -g_{\beta\mu} L^\mu{}_\alpha.$$

If we look at L with two covariant (lowered) indices, $L_{\alpha\beta} := g_{\alpha\mu} L^{\mu}{}_{\beta}$, then we have $L_{\alpha\beta} = -L_{\beta\alpha}$, or $L_{\cdot\cdot}$ is antisymmetric. They are also imaginary, because the Λ is real, and we stuck a $-i$ in the exponent[1].

(c) How many independent 4×4 imaginary antisymmetric matrices are there? Choose as a basis for this vector space (of antisymmetric imaginary 4×4 matrices) one matrix for each pair of indices α, β taking values $0, 1, 2, 3$, with $\alpha < \beta$, and define for each such pair $(\alpha\beta)$ the matrix

$$L^{(\alpha\beta)}_{\mu\nu} = i\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - i\delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}.$$

Note that matrix multiplication requires one index up and one down to be summed, so more relevant is

$$L^{(\alpha\beta)\mu}{}_{\nu} = ig^{\alpha\mu}\delta^{\beta}_{\nu} - i\delta^{\alpha}_{\nu} g^{\beta\mu},$$

where we have defined $g^{\beta\mu}$ to be the inverse of $g_{\mu\nu}$ (even though all the matrix elements are the same). With it one can raise indices $V^{\mu} = g^{\mu\nu} V_{\nu}$ on any covariant vector to make a contravariant one. Show that $\Lambda = \exp\left(-i\eta L^{(0j)}\right)$ generates a Lorentz transformation for $\mathcal{O}'$ moving to the left with velocity $v = c \tanh \eta$ in the j direction, and $\Lambda = \exp\left(-i\theta L^{(1\,2)}\right)$ generates a rotation through θ about the z axis.

(d) Show that

$$\left[L^{(\alpha\beta)}, L^{(\gamma\rho)}\right] = -ig^{\alpha\gamma}L^{(\beta\rho)} + ig^{\alpha\rho}L^{(\beta\gamma)} + ig^{\beta\gamma}L^{(\alpha\rho)} - ig^{\beta\rho}L^{(\alpha\gamma)},$$

which shows that the $L^{(\alpha\beta)}$ do form a basis for a Lie algebra, closed under commutation.

[1] Sort of so L would be hermitean, except it's not. For Euclidean space instead of Minkowski, we would have unitary Λ and hermitean L.