The gradient operator

We can define the gradient operator

$$\vec{\nabla} = \sum_i \hat{e}_i \frac{\partial}{\partial x_i}. \tag{1}$$

While this looks like an ordinary vector, the coefficients are not numbers V_i but are operators, which do not commute with functions of the coordinates x_i. We can still write out the components straightforwardly, but we must be careful to keep the order of the operators and the fields correct.

The gradient of a scalar field $\Phi(\vec{r})$ is simply evaluated by distributing the gradient operator

$$\vec{\nabla}\Phi = (\sum_i \hat{e}_i \frac{\partial}{\partial x_i})\Phi(\vec{r}) = \sum_i \hat{e}_i \frac{\partial \Phi}{\partial x_i}. \tag{2}$$

Because the individual components obey the Leibnitz rule $\frac{\partial AB}{\partial x_i} = \frac{\partial A}{\partial x_i} B + A \frac{\partial B}{\partial x_i}$, so does the gradient, so if A and B are scalar fields,

$$\vec{\nabla} AB = (\vec{\nabla} A)B + A\vec{\nabla} B. \tag{3}$$

The general application of the gradient operator $\vec{\nabla}$ to a *vector* $\vec{A}$ gives an object with coefficients with two indices, a *tensor*. Some parts of this tensor, however, can be simplified. The first (which is the trace of the tensor) is called the *divergence* of the vector, written and defined by

$$\begin{aligned}\vec{\nabla} \cdot \vec{A} &= (\sum_i \hat{e}_i \frac{\partial}{\partial x_i}) \cdot (\sum_j \hat{e}_j B_j) = \sum_{ij} \hat{e}_i \cdot \hat{e}_j \frac{\partial B_j}{\partial x_i} = \sum_{ij} \delta_{ij} \frac{\partial B_j}{\partial x_i} \\ &= \sum_i \frac{\partial B_i}{\partial x_i}. \end{aligned} \tag{4}$$

In asking about Leibnitz' rule, we must remember to apply the divergence operator only to vectors. One possibility is to apply it to the vector $\vec{V} = \Phi\vec{A}$, with components $V_i = \Phi A_i$. Thus

$$\begin{aligned}\vec{\nabla} \cdot (\Phi\vec{A}) &= \sum_i \frac{\partial(\Phi A_i)}{\partial x_i} = \sum_i \frac{\partial \Phi}{\partial x_i} A_i + \Phi \sum_i \frac{\partial A_i}{\partial x_i} \\ &= (\vec{\nabla}\Phi) \cdot \vec{A} + \Phi\vec{\nabla} \cdot \vec{A}. \end{aligned} \tag{5}$$

We could also apply the divergence to the cross product of two vectors,

$$\begin{aligned}\vec{\nabla}\cdot(\vec{A}\times\vec{B}) &= \sum_i \frac{\partial(\vec{A}\times\vec{B})_i}{\partial x_i} = \sum_i \frac{\partial(\sum_{jk}\epsilon_{ijk}A_jB_k)}{\partial x_i} = \sum_{ijk}\epsilon_{ijk}\frac{\partial(A_jB_k)}{\partial x_i} \\ &= \sum_{ijk}\epsilon_{ijk}\frac{\partial A_j}{\partial x_i}B_k + \sum_{ijk}\epsilon_{ijk}A_j\frac{\partial B_k}{\partial x_i}. \qquad (6)\end{aligned}$$

This is expressible in terms of the *curls* of $\vec{A}$ and $\vec{B}$.

The curl is like a cross product with the first vector replaced by the differential operator, so we may write the i'th component as

$$(\vec{\nabla}\times\vec{A})_i = \sum_{jk}\epsilon_{ijk}\frac{\partial}{\partial x_j}A_k. \qquad (7)$$

We see that the last expression in (6) is

$$\sum_k\Big(\sum_{ij}\epsilon_{kij}\frac{\partial A_j}{\partial x_i}\Big)B_k - \sum_j A_j\sum_{ik}\epsilon_{jik}\frac{\partial B_k}{\partial x_i} = (\vec{\nabla}\times\vec{A})\cdot\vec{B} - \vec{A}\cdot(\vec{\nabla}\times\vec{B}). \qquad (8)$$

where the sign which changed did so due to the transpositions in the indices on the ϵ, which we have done in order to put things in the form of the definition of the curl. Thus

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = (\vec{\nabla}\times\vec{A})\cdot\vec{B} - \vec{A}\cdot(\vec{\nabla}\times\vec{B}). \qquad (9)$$

Vector algebra identities apply to the curl as to any ordinary vector, except that one must be careful not to change, by reordering, what the differential operators act on. In particular, the "bac-cab" equation becomes

$$\vec{A}\times(\vec{\nabla}\times\vec{B}) = \sum_i A_i\vec{\nabla}B_i - \sum_i A_i\frac{\partial\vec{B}}{\partial x_i}. \qquad (10)$$