RU-88-010

DIFFERENTIAL FORMS AND ALL THAT

by

H. Zapolsky

We know that a tangent (contravariant) vector may be written as

$$\mathbf{v} = v^i \mathbf{e}_i \tag{1}$$

where the $\mathbf{e}_i$ are some (unit magnitude) basis vectors which live in a space "tangent" to some curve. The transformation rule for the components of a contravariant vector is:

$$v^{k'} = \frac{\partial x^{k'}}{\partial x^i} v^i \tag{2}$$

and, in order for $\mathbf{v}$ to be an invariant object (i.e., $v^{k'}\mathbf{e_{k'}} = v^i\mathbf{e_i}$), the basis vectors must transform as:

$$\mathbf{e}_{k'} = \frac{\partial x^i}{\partial x^{k'}} \mathbf{e}_i \tag{3}$$

These rules are sufficient to enable us to do all the differential geometry which we need to do, but, even with the Einstein summation convention, they put some strain on the memory. Is there a way in which we can ease this strain? The answer is, yes! We recall, from the rules of elementary calculus, that

$$\frac{\partial}{\partial x^{k'}} = \frac{\partial x^i}{\partial x^{k'}} \frac{\partial}{\partial x^i} \tag{4}$$

which is precisely the way in which the $\mathbf{e}_i$ transform. We can exploit this by writing any contravariant vector symbolically as:

$$\mathbf{v} = v^i \frac{\partial}{\partial x^i} \tag{5}$$

This notation exhibits an isomorphism between tangent vectors and derivative operators. Let us take this literally. What does the operator $\mathbf{v}$ do to some function, $f(x)$?

$$\mathbf{v}[f] = v^i \frac{\partial f}{\partial x^i} = \mathbf{v} \ \cdot \nabla f \equiv df(\mathbf{v}) \tag{6}$$

The last expression is a new bit of notation, which we will exploit in a moment. $\mathbf{v}[f]$ tells us how the function changes when we move in the direction $\mathbf{v}$ a distance $\triangle s$, where

$$(\triangle s)^2 = g_{ij} v^i v^j \tag{7}$$

Notice that $\mathbf{v}[f] \equiv df(\mathbf{v})$ is just a number (scalar) associated with the function f and the vector $\mathbf{v}$. We have written the right hand side in a form which suggests that there might be an operator df, which maps the contravariant vector $\mathbf{v}$ into a scalar. How would we write df so that it does this?

$$df = \frac{\partial f}{\partial x^i} dx^i$$

will work, because $dx^i(\mathbf{v}) \equiv \mathbf{v}[x^i] = v^j \dfrac{\partial x^i}{\partial x^j} = v^i$ and therefore

$$df(\mathbf{v}) = f_{,i}\, v^i = \mathbf{v} \cdot \nabla f \tag{8}$$

The object df is called a differential form (or 1–form, to be more precise). It is isomorphic to a covariant vector, with components $f_{,i}$. Any covariant vector, with arbitrary components u_i can be written as

$$\tilde{u} = u_i \, \tilde{d}x^i \tag{9}$$

From now on we use the tilde notation $\tilde{d}x^i$ when we are thinking of this object as a 1–form rather than an ordinary coordinate differential. As we did with ordinary tangent ("contravariant") vectors, we insist that 1–forms ("covariant" vectors) are invariant objects, in the sense that $u_{k'} \tilde{d}x^{k'} = u_i \tilde{d}x^i$. Since

$$\tilde{d}x^{k'} = \frac{\partial x^{k'}}{\partial x^i} \tilde{d}x^i,$$

invariance requires that

$$u_{k'} = \frac{\partial x^i}{\partial x^{k'}} u_i$$

which is the correct transformation rule for covariant components.

We define the exterior (or "wedge") product of two 1–forms, $\tilde{\omega}, \tilde{\sigma}$ as

$$\tilde{\omega} \wedge \tilde{\sigma} \equiv \tfrac{1}{2}[\tilde{\omega} \otimes \tilde{\sigma} - \tilde{\sigma} \otimes \tilde{\omega}] \tag{10}$$

This is called, for obvious reasons, a 2–form. Clearly

$$\tilde{\omega} \wedge \tilde{\sigma} = -\tilde{\sigma} \wedge \tilde{\omega}$$

and behaves under coordinate transformations like a doubly covariant, anti-symmetric rank 2 tensor. Since we can write

$$\tilde{\omega} = \omega_i \, \tilde{d}x^i$$
$$\tilde{\sigma} = \sigma_j \, \tilde{d}x^j$$

it is clear that any 2–form can be written as

$$\tilde{\omega} \wedge \tilde{\sigma} = \omega_i \sigma_j \, \tilde{d}x^i \wedge \tilde{d}x^j$$

Notice that terms like $\tilde{d}x^1 \wedge \tilde{d}x^1$ do not appear in this expansion because of the anti-symmetric property of the wedge product. A 3–form would look like

$$\begin{aligned}\tilde{\omega} \wedge \tilde{\sigma} \wedge \tilde{\tau} \equiv \tfrac{1}{6}[&\tilde{\omega} \otimes \tilde{\sigma} \otimes \tilde{\tau} + \tilde{\sigma} \otimes \tilde{\tau} \otimes \tilde{\omega} \\ &+ \tilde{\tau} \otimes \tilde{\omega} \otimes \tilde{\sigma} - \tilde{\omega} \otimes \tilde{\tau} \otimes \tilde{\sigma} \\ &- \tilde{\tau} \otimes \tilde{\sigma} \otimes \tilde{\omega} - \tilde{\sigma} \otimes \tilde{\omega} \otimes \tilde{\tau}]\end{aligned} \tag{11}$$

and it is clear how to extend this to n–forms.

We have seen how to generate a 1–form by differentiating a function (0– form, if you like). We generalize this notion by defining the exterior derivative, $\tilde{d}$, which operates on an n–form to generate an n+1– form. For $\tilde{\omega} = \omega_i(\mathbf{x})\tilde{d}x^i$, we write

$$\tilde{d}\tilde{\omega} \equiv \omega_{i,j}\, \tilde{d}x^j \wedge \tilde{d}x^i = -\omega_{i,j}\, \tilde{d}x^i \wedge \tilde{d}x^j \tag{12}$$

The ordering is important here. Also notice that, if we blindly applied the Liebniz rule for the differentiation of products, we would have obtained additional terms of the form $\tilde{d}(\tilde{d}x^i)$. We assert (by definition) that

$$\tilde{d}(\tilde{d}x^i) \equiv 0 \tag{13}$$

If $\tilde{\omega} = \tilde{d}f$, we say that it is an **exact** 1–form. Suppose this is the case; then

$$\begin{gathered}\tilde{\omega} = f_{,i}\, \tilde{d}x^i \\ \tilde{d}\tilde{\omega} = \tilde{d}(\tilde{d}f) = f_{,ij}\, \tilde{d}x^j \wedge \tilde{d}x^i = \sum_{j<i}[f_{,ij} - f_{,ji}]\tilde{d}x^j \wedge \tilde{d}x^i = 0\end{gathered}$$

The next to last expression comes from the anti-symmetry of the wedge product, and the null result comes from the equality of mixed partial derivatives. We can extend this to n–forms. If $\tilde{\alpha}$ is a p– form and $\tilde{\beta}$ is a q–form, then

$$\tilde{d}(\tilde{\alpha} \wedge \tilde{\beta}) \equiv \tilde{d}\tilde{\alpha} \wedge \tilde{\beta} + (-1)^p\, \tilde{\alpha} \wedge \tilde{d}\tilde{\beta} \tag{13}$$

is a p+q+1–form. The $(-1)^p$ comes from commuting the $\tilde{d}$ operator through the p differentials in $\tilde{\alpha}$. For **any** form (again, from equality of mixed partial derivatives):

$$\tilde{d}(\tilde{d}\tilde{\omega}) = 0 \tag{14}$$

This is a very compact way of writing an extremely important theorem. Suppose we write a 1–form, $\tilde{a}$, as

$$\tilde{a} = a_i\tilde{d}x^i = a_x\tilde{d}x + a_y\tilde{d}y + a_z\tilde{d}z$$

Then

$$\begin{aligned}\tilde{d}\tilde{a} &= a_{x,y}\tilde{d}y \wedge \tilde{d}x + a_{x,z}\tilde{d}z \wedge \tilde{d}x + a_{y,x}\tilde{d}x \wedge \tilde{d}y \\ &\quad + a_{y,z}\tilde{d}z \wedge \tilde{d}y + a_{z,x}\tilde{d}x \wedge \tilde{d}z + a_{z,y}\tilde{d}y \wedge \tilde{d}z \\ &= (a_{y,x} - a_{x,y})\tilde{d}x \wedge \tilde{d}y + (a_{x,z} - a_{z,x})\tilde{d}z \wedge \tilde{d}x \\ &\quad + (a_{z,y} - a_{y,z})\tilde{d}y \wedge \tilde{d}z\end{aligned}$$

If we think of the a_i as components of an ordinary (covariant) vector, then the "components" of $\tilde{d}a$ are clearly those of $\nabla \times \mathbf{a}$. If, in addition, the a_i are themselves the components of a gradient, so that $a_i = f_{,i}$, then

$$\tilde{a} = f_{,i}\,\tilde{d}x^i = \tilde{d}f$$

and

$$\tilde{d}\tilde{a} = \tilde{d}(\tilde{d}f) = 0 \Longleftrightarrow \nabla\times(\nabla f) = 0$$

Let us instead look at a 2–form, which I write in the suggestive fashion:

$$\tilde{b} = b_x\,\tilde{d}y \wedge \tilde{d}z + b_y\,\tilde{d}z \wedge \tilde{d}x + b_z\,\tilde{d}x \wedge \tilde{d}y \tag{15}$$

Its exterior derivative is

$$\begin{aligned}\tilde{d}\tilde{b} &= b_{x,x}\,\tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z + b_{y,y}\,\tilde{d}y \wedge \tilde{d}z \wedge \tilde{d}x + b_{z,z}\,\tilde{d}z \wedge \tilde{d}x \wedge \tilde{d}y \\ &= (b_{x,x} + b_{y,y} + b_{z,z})\,\tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z\end{aligned}$$

The one "component" left is clearly $\nabla \cdot \mathbf{b}$ where the vector $\mathbf{b}$ has components b_i. If, in addition, $\mathbf{b} = \nabla \times \mathbf{a}$, then $\tilde{b}$ is an **exact** 2– form.

$$\begin{aligned}\tilde{b} &= \tilde{d}\tilde{a} \\ \tilde{a} &= a_x\tilde{d}x + a_y\tilde{d}y + a_z\tilde{d}z \\ \tilde{d}\tilde{b} = \tilde{d}(\tilde{d}\tilde{a}) &= 0 \Longleftrightarrow \nabla \cdot (\nabla \times \mathbf{a}) = 0\end{aligned} \tag{16}$$

In three dimensions, we can't go any farther, since (say)

$$\tilde{d}x \wedge (\tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z) \equiv 0$$

but it should be clear that in n-dimensions, there will be n–1 identities of the form

$$\begin{aligned}\nabla \times \nabla f &= 0 \\ \nabla \cdot (\nabla \times \mathbf{a}) &= 0 \\ &\vdots\end{aligned}$$

As we shall see, the charge–current continuity equation, $\dfrac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$ is the remaining identity in a four–dimensional Minkowski space.

You should not be surprised that there are a set of closely related integral theorems. They can all be written down in one line!

$$\int_V \tilde{d}\omega = \oint_S \tilde{\omega} \tag{17}$$

where the volume of integration can have any dimensionality (up to n), and the surface which encloses it has one less dimension. This is the generalized Stokes' theorem. If $\tilde{\omega}$ is a 0–form (ordinary function) then we have:

$$\int_{\mathbf{x}_1}^{\mathbf{x}_2} \nabla f \cdot \mathbf{dl} = f(\mathbf{x}_2) - f(\mathbf{x}_1)$$

where $f(\mathbf{x})$ is any function; this is nothing more than the "fundamental theorem" of (vector) calculus.

If $\tilde{\omega}$ is a 1–form, the generalized Stokes' theorem gives:

$$\int_S (\nabla \times \mathbf{v}) \cdot \mathbf{dS} = \oint_C \mathbf{v} \cdot \mathbf{dl}$$

where $\mathbf{v}(\mathbf{x})$ is any vector field.

If $\tilde{\omega}$ is a 2–form, we get:

$$\int_V (\nabla \cdot \mathbf{v}) dV = \oint_S \mathbf{v} \cdot \mathbf{dS}$$

Clearly, in n dimensions, there are n Stokes' (Gauss') theorems.

We have seen that the operation of exterior differentiation on forms reproduces all the important vector derivative operations, ∇, $\nabla\cdot$, and $\nabla\times$. Furthermore, using this formalism, it is possible to compute the effect of these operations, almost without thinking. These computations are no more difficult in arbitrary coordinates than they are in Cartesian coordinates. Recall that

$$ds^2 = g_{ij} dx^i dx^j$$

is quadratic in the dx^i. Any such expression can always be diagonalized and normalized (recall the normal modes of classical mechanics). For example,

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \\ &\equiv (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 \end{aligned}$$

with

$$\begin{aligned} \tilde{\omega}^1 &\equiv \tilde{d}r \\ \tilde{\omega}^2 &\equiv r\tilde{d}\theta \\ \tilde{\omega}^3 &\equiv r\sin\theta\tilde{d}\phi \end{aligned} \tag{18}$$

The squares are not to be thought of as wedge products in the first expression above—we merely use that notation to define an orthonormal basis of 1–forms. Suppose we want to calculate the gradient of a function in spherical coordinates. We write

$$\tilde{d}f = f_{,i}\,\tilde{d}x^i \equiv v_i \tilde{\omega}^i$$

and the v_i will be the appropriate components of the gradient. Notice that, in general, the $\tilde{\omega}^i$ are not exact 1– forms; i.e., $r\tilde{d}\theta \neq \tilde{d}(\text{anything})$. Let's see how this works in spherical coordinates.

$$\begin{aligned}\tilde{d}f &= f_{,r}\,\tilde{d}r + f_{,\theta}\,\tilde{d}\theta + f_{,\phi}\,\tilde{d}\phi \\ &= f_{,r}\,\tilde{\omega}^1 + \frac{1}{r}f_{,\theta}\,\tilde{\omega}^2 + \frac{1}{r\sin\theta}f_{,\phi}\,\tilde{\omega}^3\end{aligned}$$

or, by inspection,

$$\begin{aligned}(\nabla f)_r &= \frac{\partial f}{\partial r} \\ (\nabla f)_\theta &= \frac{1}{r}\frac{\partial f}{\partial \theta} \\ (\nabla f)_\phi &= \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\end{aligned}$$

We recall that $\tilde{d}$(1–form) gives the curl and $\tilde{d}$(2–form) gives the divergence. Suppose we have a vector with components a_r, a_θ, and a_ϕ where $\hat{\mathbf{r}}$, $\hat{\theta}$, and $\hat{\phi}$ are a right handed triad of unit vectors—the usual ones for spherical coordinates. The corresponding 1–form is

$$\tilde{a} = a_r\,\tilde{\omega}^1 + a_\theta\,\tilde{\omega}^2 + a_\phi\,\tilde{\omega}^3$$

and

$$\begin{aligned}\tilde{d}\tilde{a} = a_{r,\theta}\,\tilde{d}\theta\wedge\tilde{\omega}^1 + a_{r,\phi}\,\tilde{d}\phi\wedge\tilde{\omega}^1 + a_{\theta,r}\,\tilde{d}r\wedge\tilde{\omega}^2 \\ + a_{\theta,\phi}\,\tilde{d}\phi\wedge\tilde{\omega}^2 + a_{\phi,r}\,\tilde{d}r\wedge\tilde{\omega}^3 + a_{\phi,\theta}\,\tilde{d}\theta\wedge\tilde{\omega}^3 \\ + a_r\,\tilde{d}\omega^1 + a_\theta\,\tilde{d}\omega^2 + a_\phi\,\tilde{d}\omega^3\end{aligned}$$

where, from Eq.(18),

$$\begin{aligned}\tilde{d}r &= \tilde{\omega}^1 \\ \tilde{d}\theta &= \frac{\tilde{\omega}^2}{r} \\ \tilde{d}\phi &= \frac{\tilde{\omega}^3}{r\sin\theta} \\ \tilde{d}\tilde{\omega}^1 &= \tilde{d}(\tilde{d}r) = 0 \\ \tilde{d}\tilde{\omega}^2 &= \tilde{d}r\wedge\tilde{d}\theta = \frac{\tilde{\omega}^1\wedge\tilde{\omega}^2}{r} \\ \tilde{d}\tilde{\omega}^3 &= \sin\theta\,\tilde{d}r\wedge\tilde{d}\phi + r\cos\theta\,\tilde{d}\theta\wedge\tilde{d}\phi \\ &= \frac{\tilde{\omega}^1\wedge\tilde{\omega}^3}{r} + \frac{cot\theta\,\tilde{\omega}^2\wedge\tilde{\omega}^3}{r}\end{aligned}$$

Grouping terms, we have:

$$\begin{aligned}\tilde{d}\tilde{a} = \left(\frac{a_{\phi,\theta}}{r} - \frac{a_{\theta,\phi}}{r\sin\theta} + \frac{a_\phi\ \cot\theta}{r}\right)\tilde{\omega}^2\wedge\tilde{\omega}^3 \\ + \left(\frac{a_{r,\phi}}{r\sin\theta} - a_{\phi,r} - \frac{a_\phi}{r}\right)\tilde{\omega}^3\wedge\tilde{\omega}^1 \\ + \left(a_{\theta,r} - \frac{a_{r,\theta}}{r} + \frac{a_\theta}{r}\right)\tilde{\omega}^1\wedge\tilde{\omega}^2\end{aligned} \tag{19}$$

The coefficients of $\tilde{\omega}^2 \wedge \tilde{\omega}^3$, $\tilde{\omega}^3 \wedge \tilde{\omega}^1$, and $\tilde{\omega}^1 \wedge \tilde{\omega}^2$ in the above expression are the respective $\hat{\mathbf{r}}$, $\hat{\theta}$, and $\hat{\phi}$ components of $\nabla \times \mathbf{a}$. (Note the cyclic ordering of the basis 1–forms in the wedge products). You will not be able to appreciate the value of using differential forms to make this computation unless you have tried to do it in the more standard way!

There are two more operations on forms which we wish to introduce. The first of these is the **norm** operation, which is related to taking the modulus of a tensor in the standard notation. Any p–form can be written as:

$$\tilde{a} = a_{|i_1 i_2 \ldots i_p|}\, \tilde{\omega}^{i_1} \wedge \tilde{\omega}^{i_2} \wedge \cdots \wedge \tilde{\omega}^{i_p} \tag{20}$$

where the vertical bars mean that $i_1 < i_2 < \cdots < i_p$. For example,

$$\tilde{a} = a_{12}\, \tilde{\omega}^1 \wedge \tilde{\omega}^2 + a_{21}\, \tilde{\omega}^2 \wedge \tilde{\omega}^1 = (a_{12} - a_{21})\, \tilde{\omega}^1 \wedge \tilde{\omega}^2 \equiv a_{|12|}\, \tilde{\omega}^1 \wedge \tilde{\omega}^2$$

The square of the norm of $\tilde{a}$ is then

$$\|\tilde{a}\|^2 \equiv a_{|i_1 i_2 \ldots i_p|}\, a^{i_1 i_2 \ldots i_p} \tag{20}$$

where the $a^{i_1 i_2 \ldots i_p}$ are the fully contravariant components of the rank p covariant tensor represented by the p–form $\tilde{a}$.

The **dual** of a p–form is an (n–p) form, with (almost) the same components. It is defined by the "star" operation:

$$({}^*\tilde{a})_{k_1 k_2 \ldots k_{n-p}} \equiv a^{|i_1 i_2 \ldots i_p|} \epsilon_{i_1 i_2 \ldots i_p k_1 k_2 \ldots k_{n-p}} \tag{21}$$

where $\epsilon_{i_1 i_2 \ldots i_n}$ is the fully anti-symmetric Levi-Civita symbol in n dimensions. In **E3**, this is particularly simple. Suppose we have a 1–form:

$$\tilde{a} = a_1\, \tilde{dx} + a_2\, \tilde{dy} + a_3\, \tilde{dz}$$

then,

$${}^*\tilde{a} = a_1\, \tilde{dy} \wedge \tilde{dz} + a_2\, \tilde{dz} \wedge \tilde{dx} + a_3\, \tilde{dx} \wedge \tilde{dy}$$

A little more care is necessary in **M4**, since there are sign changes in going from covariant to contravariant components. Thus, if

$$\tilde{a} = a_0\, \tilde{dt} + a_1\, \tilde{dx} + a_2\, \tilde{dy} + a_3\, \tilde{dz}$$

we get

$${}^*\tilde{a} = -a_0\, \tilde{dx} \wedge \tilde{dy} \wedge \tilde{dz} - a_1\, \tilde{dy} \wedge \tilde{dz} \wedge \tilde{dt} + a_2\, \tilde{dz} \wedge \tilde{dt} \wedge \tilde{dx} - a_3\, \tilde{dt} \wedge \tilde{dx} \wedge \tilde{dy}$$

(where we have adopted the convention that $\epsilon^{0123} = -\epsilon_{0123} = +1$). Look at the third term. $a^2 = -a_2$, but $\epsilon_{2310} = -\epsilon_{1230} = +\epsilon_{0123} = -1$, so things are a bit tricky. As another example, suppose we have a 2–form in **M4**:

$$\tilde{\sigma} \equiv \sigma_{tr}\, \tilde{dt} \wedge \tilde{dr} = \sigma_{tr} \tilde{\omega}^0 \wedge \tilde{\omega}^1$$
$${}^*\tilde{\sigma} = \sigma^{tr} \epsilon_{0123}\, \tilde{\omega}^2 \wedge \tilde{\omega}^3 = \sigma_{tr}\, r^2\, \sin\theta\, \tilde{d\theta} \wedge \tilde{d\phi}$$

All of E&M can now be written in a few lines. The differential form which corresponds to the 4–potential, A^μ, is

$$\tilde{A} = \phi\,\tilde{d}t - A_x\,\tilde{d}x - A_y\,\tilde{d}y - A_z\,\tilde{d}z \tag{21}$$

(Notice that we must use the **co**variant components of **A** to define the 1–form). There is then a natural 2–form which reproduces the doubly covariant components of the electromagnetic field tensor;

$$\begin{aligned}\tilde{F} \equiv \tilde{d}A = E_x\,\tilde{d}t\wedge\tilde{d}x + E_y\,\tilde{d}t\wedge\tilde{d}y + E_z\,\tilde{d}t\wedge\tilde{d}z \\ - B_x\,\tilde{d}y\wedge\tilde{d}z - B_y\,\tilde{d}z\wedge\tilde{d}x - B_z\,\tilde{d}x\wedge\tilde{d}y\end{aligned} \tag{22}$$

Gauge transformations are given by $\tilde{A}' = \tilde{A} + \tilde{d}\psi$, where ψ is an arbitrary function; $\psi = \psi(\mathbf{x}, t)$. This leaves the field invariant, since $\tilde{d}(\tilde{d}\psi) \equiv 0$. Two of the Maxwell equations are given by:

$$\begin{gathered}\tilde{d}F = \tilde{d}(\tilde{d}\tilde{A}) \equiv 0 \\ \Updownarrow \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}\end{gathered} \tag{23}$$

The remaining two equations relate the fields to the sources, so we must use the charge-current 1–form:

$$\tilde{J} \equiv \rho\,\tilde{d}t - J_x\,\tilde{d}x - J_y\,\tilde{d}y - J_z\,\tilde{d}z \tag{24}$$

Now,

$$^*\tilde{J} = -\rho\,\tilde{d}x\wedge\tilde{d}y\wedge\tilde{d}z + J_x\,\tilde{d}t\wedge\tilde{d}y\wedge\tilde{d}z - J_y\,\tilde{d}t\wedge\tilde{d}x\wedge\tilde{d}z + J_z\,\tilde{d}t\wedge\tilde{d}x\wedge\tilde{d}y$$

and

$$\tilde{d}^*\tilde{J} = -\left(\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J}\right)\tilde{d}t\wedge\tilde{d}x\wedge\tilde{d}y\wedge\tilde{d}z \equiv 0 \tag{25}$$

from charge conservation. This suggests that $^*\tilde{J}$ is an **exact** 3–form (but, see the addendum to these notes). Since $\tilde{d}(\tilde{d}F) = \tilde{d}(\tilde{d}(\tilde{d}A)) \equiv 0$, the only other exact 3–form available to us is $\tilde{d}^*\tilde{d}\tilde{F}$, so we conclude that:

$$\tilde{d}^*\tilde{d}\tilde{F} \propto^* \tilde{J} \tag{26}$$

With the constant of proportionality in Eq.(26) taken as -4π (gaussian units), the above equation is equivalent to

$$\begin{gathered}\nabla\cdot\mathbf{E} = 4\pi\rho \\ \nabla\times\mathbf{B} = 4\pi\mathbf{J} + \frac{\partial\mathbf{E}}{\partial t}\end{gathered} \tag{27}$$

In this notation, the field invariants are

$$\begin{gathered}\|\tilde{F}\|^2 = B^2 - E^2 \\ \text{and} \\ (\tilde{F})_{\alpha\beta}(^*\tilde{F})^{\alpha\beta} = 2\mathbf{E}.\mathbf{B}\end{gathered} \tag{28}$$

In the above, we have set c=1. In order to reclaim the proper factors of c, you need only let $t \longrightarrow ct$, $\mathbf{J} \longrightarrow \mathbf{J}/c$, and $\mathbf{A} \longrightarrow \mathbf{A}/c$ after the form computations are performed.

You may feel that all of this is somewhat artificial, and not worth the effort just to gain some additional compactness in the way we write our equations. If that were the only reason for introducing differential forms, I would agree with you. But, by freeing us from coordinate bases (remember, the $\tilde{\omega}^i$ are not, in general, derivatives of coordinates), the formalism makes it possible to compute in a much more efficient fashion. In particular, by avoiding the general tensor transformation formulae, it assures that we will never make a lengthy calculation of some tensor component that turns out to be zero. (Review the computation of $\nabla \times \mathbf{a}$ in spherical polar coordinates, given above). As another example of this convenience, consider the Lorentz transformation of the $\mathbf{E}$ and $\mathbf{B}$ fields from a frame $\mathbf{x}, t$ to a frame $\mathbf{x}', t'$ with

$$\begin{aligned}
x &= \gamma(x' + \beta t') \\
t &= \gamma(t' + \beta x') \\
y &= y' \\
z &= z'
\end{aligned}$$

Rewriting Eq.(22) in terms of the primed coordinate differentials,

$$\begin{aligned}
\tilde{F} &= \gamma^2 E_x(\tilde{d}t' + \beta \tilde{d}x') \wedge (\tilde{d}x' + \beta \tilde{d}t') \\
&\quad + \gamma(\tilde{d}t' + \beta \tilde{d}x')(E_y\, \tilde{d}y' + E_z\, \tilde{d}z') \\
&\quad - B_x\, \tilde{d}y' \wedge \tilde{d}z' - \gamma B_y\, \tilde{d}z' \wedge (\tilde{d}x' + \beta \tilde{d}t') \\
&\quad - \gamma B_z\, (\tilde{d}x' + \beta \tilde{d}t') \wedge \tilde{d}y' \\
&= \gamma^2(1-\beta^2) E_x\, \tilde{d}t' \wedge \tilde{d}x' + \gamma(E_y - \beta B_z)\, \tilde{d}t' \wedge \tilde{d}y' \\
&\quad + \gamma(E_z + \beta B_y)\, \tilde{d}t' \wedge \tilde{d}z' - B_x\, \tilde{d}y' \wedge \tilde{d}z' \\
&\quad - \gamma(B_y + \beta E_z)\, \tilde{d}z' \wedge \tilde{d}x' - \gamma(B_z - \beta E_y)\, \tilde{d}x' \wedge \tilde{d}y'
\end{aligned}$$

and we can immediately read off the components of $\mathbf{E}'$ and $\mathbf{B}'$ according to Eq.(22). Since $\gamma^2(1-\beta^2) \equiv 1$ we have

$$\begin{aligned}
E_{x'} &= E_x \\
E_{y'} &= \gamma(E_y - \beta B_z) \\
E_{z'} &= \gamma(E_z + \beta B_y) \\
B_{x'} &= B_x \\
B_{y'} &= \gamma(B_y + \beta E_z) \\
B_{z'} &= \gamma(B_z - \beta E_y)
\end{aligned}$$

In short, this elegant formalism of differential forms, invented over fifty years ago by Élie Cartan, is beginning to make its impact in Physics. In fact, in some areas like General Relativity, it makes possible computations by hand which would never otherwise be attempted by a sensible physicist— even a theorist! For those of you who wish to pursue this subject further, I recommend the following references:

"Gravitation", Misner, Thorne, and Wheeler, (W. H. Freeman, 1973)
"Differential Forms", H. Flanders, (Academic Press, 1963)
"Differential Geometry", B. O'Neill, (Academic Press, 1966)

Addendum to "Forms and All That"

In "deriving" Maxwell's source equations, I have argued that, since $\tilde{d}^*\tilde{J} = 0$ (from charge conservation), $^*J = \tilde{d}\tilde{\omega}$, where $\tilde{\omega}$ is a 2–form, and the only non-trivial 2–form we have left is $^*\tilde{F}$. In other words, I have *implicitly* assumed that

$$\tilde{d}\tilde{\sigma} = 0 \Longrightarrow \tilde{\sigma} = \tilde{d}\tilde{\omega}$$

Now, the converse of this theorem is always true (it involves nothing more than the equality of mixed partial derivatives), but the theorem itself is only true for spaces of "simple" topology, where closed curves (or surfaces, etc.) can be continuously deformed to a point. To see how this works, look at the 1–form

$$\tilde{\omega} \equiv -\frac{y}{r^2}\,\tilde{d}x + \frac{x}{r^2}\,\tilde{d}y \qquad \text{with} \qquad r^2 \equiv x^2 + y^2$$

For this example $\tilde{d}\tilde{\omega} = 0$ and we might be tempted to write

$$\tilde{\omega} = \tilde{d}\eta \qquad \text{with} \qquad \eta = \arctan\frac{y}{x}$$

which works "almost" everywhere, although it is ill defined at $x = y = 0$. A little more care is needed here. In the first place, our 1– form is singular at the origin, so, in order to be on the safe side, we should exclude that point from our space. This changes the topology of the space, since any closed curve which surrounds the origin can no longer be continuously contracted to a point. (An alternative way of saying this is that a curve connecting two points in our space cannot be continuously deformed into any other arbitrary curve connecting the same two points; it can only be deformed into another curve which lies on the "same side" of the singularity). This is the underlying reason for the failure of our "theorem" above. We can restore a simple topology, however, by exluding not only the origin, but also the positive x axis. Now we are forbidden from drawing closed curves which circle the origin. (If this sounds reminiscent of the theory of complex variables, I assure you that it is intimately connected!). Now the arc-tangent is well defined everywhere in our "manifold", and things work smoothly.

This is all very relevant to the magnetic monopole problem. We have "derived" the Maxwell theory for a simply connected space-time. In such a space-time,

$$\nabla \cdot \mathbf{B} = 0 \Longrightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

and

$$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$$

If we place a monopole at the origin, (say), we should probably exclude that point, but once we do so, we cannot contract closed surfaces which surround it to a point. We could also decide to exclude a line running from the origin to infinity, which would then prevent us from drawing such surfaces. Suppose we take a straight line going through the south–pole. Now we can write for our vector potential,

$$A_\theta = A_r = 0$$
$$A_\phi = \frac{g}{r\sin\theta}(1-\cos\theta)$$
$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{g}{r^2}\hat{\mathbf{r}}$$

If we now take a surface which almost surrounds the monopole, but with the south–pole excluded, we get

$$\int_S \mathbf{B}\cdot\mathbf{dS} = 4\pi g$$

This surface has a one-dimensional boundary, Γ, which is an infinitesimal circle surrounding the exluded south– pole. From the generalized Stokes' theorem, we expect that

$$\int_S \mathbf{B}\cdot\mathbf{dS} = \oint_\Gamma \mathbf{A}\cdot\mathbf{dl}$$

and, since the radius of the circle is $r\sin\theta$, we have, explicitly,

$$\oint_\Gamma \mathbf{A}\cdot\mathbf{dl} = \frac{g}{r\sin\theta}(1-\cos\theta)\cdot 2\pi r\sin\theta \to 4\pi g \qquad \text{as} \quad \theta\to\pi$$

The "Dirac string" is nothing more than our exluded line, and we see that the theory can accomodate monopoles if we merely make a slight change in the topology of space-time. The theory still remains deeply geometrical in character, which is the prime motivation for introducing monopoles in this particular way.

In technical terms, we have said that if $\tilde{\omega} = \tilde{d}\tilde{\sigma}$, then $\tilde{\omega}$ is an **exact** form. We also say that if $\tilde{d}\tilde{\omega} = 0$ it is a **closed** form. While it is true that all exact forms are closed, we have seen that it is not always true that all closed forms are exact. When we are dealing with non-simply connected topologies, they won't be, unless we fiddle a little bit. The formal analysis of such topologies involves a branch of mathematics known as "homology theory", and it has found some interesting recent applications in both condensed matter theory (dislocations in ordered media) and particle theory (superstrings). The interested reader may wish to refer to:

"Topology and Geometry for Physicists", C. Nash and S. Sen, (Academic Press, 1983)