On Indices and Arguments

Joel A. Shapiro

October 6, 2006

When one gets to advanced courses in Physics, notation is used in a rather sophisticated way to convey things efficiently, but this notation can be confusing at first. In particular, indices on quantities, especially vectors and tensors, and also arguments of functions, are used with several different meanings.

First let's consider indices. Their use can be categorized into four modes, which I will call **specific**, **free**, **dummy**, and **indicative**. For example, if I say the first Pauli spin matrix is given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the index 1 on the σ is a **specific** index, telling something about only one component of the vector $\vec{\sigma}$. Similarly, I might say for the gravitational force,

$$F_x = 0, \quad F_y = 0, \quad F_z = -mg,$$

in which each index is specific.

But if we write out the vector equation $\vec{L} = \vec{r} \times \vec{p}$ as

$$L_i = \sum_{jk} \epsilon_{ijk} r_j p_k, \tag{1}$$

the index i is a **free** index, which can take any value (of the suitable type, here 1, 2, 3 or x, y, z. The equation is supposed to be true for **each** possible value of the index. Of course, the same value must be substituted for i as the first index of the ϵ.

The indices j and k in Eq. (1) are not free, but **dummy** indices. These are summed over specific values. This is the same role as x plays in $\int_a^b f(x)\,dx$. Physicists often use, especially in relativistic contexts, the "Einstein summation convention" that says that repeated indices in a term should be summed over even if the summation sign is not explicitly written, so one might write Eq. (1) as $L_i = \epsilon_{ijk} r_j p_k$. This makes the dummy indices a bit less obvious, but one soon learns to deal with that.

Of course the summation convention does not apply to indices repeated in different terms, as for example in $J_k = L_k + S_k$ giving the total angular momentum as the sum of orbital and spin angular momentum. In that equation, k is a free index.

In relativity, the summation convention is usually restricted to a pair of the same index, once appearing covariantly and once contravariantly, as in $p^2 = p^\mu p_\mu$.

The fourth mode of use is **indicative**. When we write $\phi(x^\mu)$, we probably mean only to indicate that ϕ is a function of spacetime, not that the index μ is either free or summed over. Similarly, we might write the generic form of a lagrangian as $L(q_i, \dot{q}_i, t)$, though we do not mean that the Lagrangian only depends on one degree of freedom. (It might have been more accurate to write $L(\{q\}, \{\dot{q}\}, t)$, but we rarely do that.)

The indicative mode is often used in arguments of functions. When we talk about a function $f(x)$, we really mean the function f, not the single value which f takes when applied to a specific argument x, though strictly speaking that is what $f(x)$ means.

Things to watch out for

It is important to realize that the scope of a free or dummy variable is determined in context, and if you combine expressions you must take care to keep the appropriate variables distinct. For example, in quantum mechanics the orbital angular momentum from Eq. (1) becomes an operator

$$L_j = -i\epsilon_{jk\ell} r_k \frac{\partial}{\partial x_\ell}$$

If we are asked to find the commutator with the position $\vec{r}$, we would be making a mistake if we wrote

$$[L_j, r_k] = \left[-i\epsilon_{jk\ell} r_k \frac{\partial}{\partial r_\ell}, r_k\right] \quad \left(= -i\epsilon_{jk\ell} r_k \left[\frac{\partial}{\partial r_\ell}, r_k\right] = -i\epsilon_{jk\ell} r_k \delta_{\ell k} = -i\epsilon_{jk\ell} r_\ell\right)$$

because the first and fourth k indices are free indices having nothing to do with the second and third k indices, which are dummies. The correct thing to do is to replace the dummy index k in the definition of L with another, so

$$\begin{aligned}
[L_j, r_k] &= \left[-i\epsilon_{jq\ell} r_q \frac{\partial}{\partial r_\ell}, r_k\right] = -i\epsilon_{jq\ell} r_q \left[\frac{\partial}{\partial r_\ell}, r_k\right] \\
&= -i\epsilon_{jq\ell} r_q \delta_{\ell k} = -i\epsilon_{jqk} r_q \\
&= i\epsilon_{jkq} r_q.
\end{aligned}$$

Notice the correct answer differs by a sign from the earlier incorrect derivation.

When we multiply together two expressions each defined as a sum, it is important to keep the summation indices distinct. This is true even for integration variables, so for example, if $f(t)$ and $g(t)$ are defined in terms of their Fourier transforms,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} d\omega, \quad g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}(\omega) e^{-i\omega t} d\omega,$$

the product is

$$f(t)g(t) \neq \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)\tilde{g}(\omega)\, e^{-i\omega t} d\omega,$$

or any other single integral, but rather as

$$f(t)g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega'\, e^{-i(\omega+\omega')t} \tilde{f}(\omega)\tilde{g}(\omega').$$

A good place to practice working with indicies is to reproduce the derived formulas from my notes on "ϵ_{ijk} and cross products in 3-D Euclidean space" (`http://www.physics.rutgers.edu/grad/615/lects/eps3deuc.pdf`).