Properties of Determinants

If M is an $N \times N$ matrix, the determinant $\det M$ is best defined as

$$\det M = \sum_P \epsilon_{P(1),...P(N)} \prod_{i=1}^{N} M_{i,P(i)}, \tag{1}$$

where the sum is over all permutations of N objects, and $\epsilon_{P(1),...P(N)} =$ sign of permutation P. Whatever definition you may know, it is easy to show this one satifies it. For example, if the matrix C has the columns of M permuted by some other permutation Q, so $C_{Q(j),k} = M_{j,k}$, then the determinant

$$\begin{aligned}\det C &= \sum_P \epsilon_{P(1),...P(N)} \prod_{i=1}^{N} C_{i,P(i)} \\ &= \sum_P \mathrm{sign}(P) \prod_{i=1}^{N} M_{Q^{-1}(i),P(i)}\end{aligned}$$

The product here is commutative and the dummy index i can be written as $i = Q(\ell)$, so a product over i is the same as one over ℓ. Also, summing over P is the same as summing over all permutations R, where $R = PQ$, and $\mathrm{sign}(P) = \mathrm{sign}(R)/\mathrm{sign}(Q) = \mathrm{sign}(R)\mathrm{sign}(Q)$, so

$$\begin{aligned}\det C &= \sum_P \mathrm{sign}(P) \prod_{\ell=1}^{N} M_{\ell,P(Q(\ell))} \\ &= \sum_R \mathrm{sign}(R)\mathrm{sign}(Q) \prod_{\ell=1}^{N} M_{\ell,R(\ell)} \\ &= \mathrm{sign}(Q) \det M.\end{aligned}$$

Thus the determinant, for example, changes sign if two columns are interchanged.

The ϵ symbol is defined so that if $(k_1, ..., k_N)$ is not a permutation of $(1, ..., N)$, then $\epsilon_{(k_1,...,k_N)} = 0$. Thus we can write eq. 1 as

$$\det M = \sum_{k_1,...,k_N} \epsilon_{k_1,...,k_N} \prod_{i=1}^{N} M_{i,k_i}, \tag{2}$$

where the sum is over all N indices, each going from 1 to N. Because each permutation of $(1, ..., N)$ will appear just once in this sum, and because all

other terms in the sum are zero, this sum is the same as in (1). If the first indices appear in some other order, as we have seen we find

$$\sum_{k_1,...,k_N} \epsilon_{k_1,...,k_N} \prod_{i=1}^{N} M_{j_i,k_i} = \det M \epsilon_{j_1,...,j_N}, \tag{3}$$

where in particular if two of the j_i's are the same, $j_a = j_b$, the left hand side vanishes by pairwise canceling of terms in the sum which interchange k_a and k_b.

If C and M are $N \times N$ matrices,

$$\begin{aligned}\det(CM) &= \sum_{k_1,...,k_N} \epsilon_{k_1,...,k_N} \prod_{i=1}^{N} (CM)_{i,k_i} \\ &= \sum_{k_1,...,k_N} \epsilon_{k_1,...,k_N} \prod_{i=1}^{N} \left(\sum_{j_i} C_{i,j_i} M_{j_i,k_i} \right) \\ &= \sum_{j_1,...,j_N} \left(\prod_{i=1}^{N} C_{i,j_i} \right) \sum_{k_1,...,k_N} \epsilon_{k_1,...,k_N} \prod_{i=1}^{N} (M_{j_i,k_i}) \\ &= \sum_{j_1,...,j_N} \left(\prod_{i=1}^{N} C_{i,j_i} \right) \epsilon_{j_1,...,j_N} \det M \\ &= \det C \det M\end{aligned}$$

so

$$\det(CM) = \det C \det M. \tag{4}$$