Geodesics in Riemannian Space

We saw that in generalized coordinates, even if we restrict our point transformations to be time-independent, the kinetic energy is in general given by a more complicated quadratic in the velocities,

$$T = \frac{1}{2} \sum_{jk} g_{jk}(\{x\}) \dot{x}_j \dot{x}_k,$$

where the mass matrix $g_{jk}(\{x\})$ can be a function on coordiate space $\{x\}$, and can have off-diagonal elements, though it is a real symmetric matrix. We can think of g as providing a metric, a measure on infinitesimal displacements dx_i

$$(ds)^2 = \sum_{jk} g_{jk}(\{x_i\}) dx_j dx_k.$$

Aside from weighting the distance each particle moves by its mass, this also allows for distances to be described appropriately for non-cartesian coordinates.

Consider a system with no forces, no potential. Then the action is just (half) the "distance" as defined by the metric g, so we expect the path to be of minimum length, to be a "straight line". What does that mean if the space is not Euclidean?

If a path in $\{x\}$ space is given by $x_i(\lambda)$, the length of the path is

$$\ell = \int_{\lambda_i}^{\lambda_f} \sqrt{(ds)^2} = \int_{\lambda_i}^{\lambda_f} \sqrt{\sum_{jk} g_{jk}(\{x_i\}) \frac{dx_j}{d\lambda} \frac{dx_k}{d\lambda}}\, d\lambda.$$

This is like Hamilton with $L \to f = \sqrt{\sum_{jk} g_{jk}(\{x_i\}) \dot{x}_j \dot{x}_k}$, with $t \to \lambda$. Then the shortest length is a stationary action, given by the Lagrange equations based on

$$f(\{x_j\}, \{\dot{x}_k\}) = \sqrt{\sum_{jk} g_{jk} \dot{x}_j \dot{x}_k},$$

where $\dot{x}_j := dx_j/d\lambda$, not the time derivative.

(a) Thus

$$\frac{\partial f}{\partial \dot{x}_i} = \frac{\sum_k g_{ik} \dot{x}_k}{\sqrt{\sum_{jk} g_{jk} \dot{x}_j \dot{x}_k}}$$

while

$$\frac{\partial f}{\partial x_i} = \frac{1}{2}\sum_{jk}\frac{\partial g_{jk}}{\partial x_i}\dot{x}_j\dot{x}_k \Bigg/ \sqrt{\sum_{jk} g_{jk}\dot{x}_j\dot{x}_k}.$$

We notice that life would be a lot simpler if we could assume $\sum_{jk} g_{jk}\dot{x}_j\dot{x}_k = 1$. We will do so later, after having justified it, but for now we just plod along.

Lagrange's equations give

$$\begin{aligned}
0 &= \frac{d}{d\lambda}\frac{\partial f}{\partial \dot{x}_i} - \frac{\partial f}{\partial x_i} \\
&= \frac{\displaystyle\sum_{jk}\frac{\partial g_{ik}}{\partial x_j}\dot{x}_j\dot{x}_k + \sum_k g_{ik}\ddot{x}_k}{\sqrt{\sum_{jk} g_{jk}\dot{x}_j\dot{x}_k}} \\
&\qquad -\frac{1}{2}\frac{\displaystyle\left(\sum_k g_{ik}\dot{x}_k\right)\left(\sum_{jk} 2g_{jk}\dot{x}_j\ddot{x}_k + \sum_{jkm}\frac{\partial g_{jk}}{\partial x_m}\dot{x}_j\dot{x}_k\dot{x}_m\right)}{\left(\sum_{jk} g_{jk}\dot{x}_j\dot{x}_k\right)^{3/2}} \\
&\qquad -\frac{1}{2}\frac{\displaystyle\sum_{jk}\frac{\partial g_{jk}}{\partial x_i}\dot{x}_j\dot{x}_k}{\sqrt{\sum_{jk} g_{jk}\dot{x}_j\dot{x}_k}}.
\end{aligned}$$

Multiplying by $\left(\sum_{mn} g_{mn}\dot{x}_m\dot{x}_n\right)^{3/2}$, we have

$$\begin{aligned}
0 &= \sum_{jkmn}\left(g_{mn}\frac{\partial g_{ik}}{\partial x_j} - \frac{1}{2}g_{in}\frac{\partial g_{jk}}{\partial x_m} - \frac{1}{2}g_{mn}\frac{\partial g_{jk}}{\partial x_i}\right)\dot{x}_j\dot{x}_k\dot{x}_m\dot{x}_n \\
&\qquad + \sum_{kmn}\left(g_{mn}g_{ik} - g_{im}g_{nk}\right)\dot{x}_m\dot{x}_n\ddot{x}_k.
\end{aligned}$$

We seem to have three differential equations for our three functions $x_i(\lambda)$, but if we multiply by $\dot{x}_i$ and sum on i, we get an identity, because the g factors in parentheses vanish when contracted with expressions symmetric under $i \leftrightarrow j$, under $j \leftrightarrow m$, and under $i \leftrightarrow n$. So we see the three equations are not independent. Why?

(b) The length has been written in a form independant of the variable used to describe the position along the path, as can be seem by the chain rule, as $\sqrt{\sum g_{jk}\frac{\partial x_j}{\partial \sigma}\frac{\partial x_k}{\partial \sigma}} = \frac{d\lambda}{d\sigma}\sqrt{\sum g_{jk}\frac{\partial x_j}{\partial \lambda}\frac{\partial x_k}{\partial \lambda}}$. But if $m = \lambda + \delta\lambda$, $\delta x_i = \dot{x}_i\delta\lambda$, so $\dot{x}_i$ times the variation due to δx_i gives zero for any path.

We may use this independence of parameterization to justify taking our parameter λ to be the distance s from the beginning up to the point in question, in which case $(d\lambda)^2 = \sum_{jk} g_{jk} dx_j dx_k$ and $\sum g_{jk} \frac{\partial x_j}{\partial \lambda} \frac{\partial x_k}{\partial \lambda} = 1$. Thus we can ignore this denominator in our Lagrange equation, and get

$$0 = \frac{d}{ds} \sum_k g_{ik} \dot{x}_k - \frac{1}{2} \sum_{jk} \frac{\partial g_{jk}}{\partial x_i} \dot{x}_j \dot{x}_k = \sum_k g_{ik} \ddot{x}_k + \sum_{jk} \left(\frac{\partial g_{ik}}{\partial x_j} - \frac{1}{2} \frac{\partial g_{jk}}{\partial x_i} \right) \dot{x}_j \dot{x}_k.$$

To extract the equations with individual $d^2 x_k / ds^2$, define $G_{\ell i}$ to be the *inverse matrix* to g_{ik}, or more precisely, because we are talking about matrices and not their matrix elements, $G = g^{-1}$. Also notice that, because it is multiplied by $\dot{x}_j \dot{x}_k$, we can replace the $\frac{\partial g_{ik}}{\partial x_j}$ in the second term with $\frac{1}{2} \frac{\partial g_{ik}}{\partial x_j} + \frac{1}{2} \frac{\partial g_{ij}}{\partial x_k}$ so we find the *geodesic* equation

$$\frac{d^2 x_i}{ds^2} + \sum_{jk} \Gamma^i{}_{jk} \frac{dx_j}{ds} \frac{dx_k}{d\tau} = 0, \quad \text{with} \quad \Gamma^i{}_{jk} := \frac{1}{2} \sum_m G_{im} \left(\frac{\partial g_{mk}}{\partial x_j} + \frac{\partial g_{mj}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_m} \right).$$

Generalized to four-dimensional space with the appropriate generalization of the Minkowski metcic, $\Gamma^\lambda{}_{\mu\nu}$ is called the **Christoffel symbol** or **affine connection**.