Response to Prof. Lovelace's complaints about problem 5.3

Prof. Lovelace has repeated complained that the problem asking for the derivation of the equations for $dr/d\tau$ and $d\theta/d\tau$ and the precession of the perihelion of Mercury are inappropriate and my solution invalid, principally because I did not justify the claim that

$$\sum_{\mu\nu} g_{\mu\nu}(x^\rho)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = c^2,$$

and that without that simplification, the Lagrange equations are very complicated.

Let me repeat the argument:

Any path $x^\mu(\lambda)$, say on $\lambda \in [0,1]$, which starts at x_I^μ at $\lambda = 0$ and ends at x_F^μ at $\lambda = 1$ and which extremizes the action

$$S = \int d\lambda \mathcal{L} \quad \text{with} \quad \mathcal{L} = mc\sqrt{\sum_{\mu\nu} g_{\mu\nu}(x^\rho)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}},$$

will satisfy the Lagrange equations

$$\begin{aligned}\frac{d}{d\lambda}&\left(mc\frac{\sum_\nu g_{\mu\nu}(x)\,dx^\nu/d\lambda}{\sqrt{\sum_{\alpha\beta} g_{\alpha\beta}(x)\,dx^\alpha/d\lambda\,dx^\beta/d\lambda}}\right)\\ &= mc\sum_{\alpha\beta}\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha/d\lambda\,dx^\beta/d\lambda}{\sqrt{\sum g_{\mu\nu}\,dx^\mu/d\lambda\,dx^\nu/d\lambda}}. \end{aligned} \tag{1}$$

This equation may be messy to solve, and in fact the solutions are not unique, but it does have solutions, and for any such solution, we can extract the necessary information, that is, the path through space-time, by observing that the equations are form-invariant under any monotonic change in parameters. Most useful is to define the proper time parameter

$$\tau(\lambda) = \frac{1}{c}\int_0^\lambda \sqrt{\sum g_{\mu\nu}\frac{dx^\mu}{d\lambda}(\lambda')\frac{dx^\nu}{d\lambda}(\lambda')}\,d\lambda',$$

for which $\dfrac{d\tau}{d\lambda} = \dfrac{1}{c}\sqrt{\sum g_{\mu\nu}\dfrac{dx^\mu}{d\lambda}\dfrac{dx^\nu}{d\lambda}}$. The same path through space-time described by $x^\mu(\lambda)$ is also described by $\tilde{x}^\mu(\tau) := x^\mu(\lambda(\tau))$, and using the chain rule,

$$\frac{d}{d\lambda} = \frac{d\tau}{d\lambda}\frac{d}{d\tau}, \qquad \frac{dx^\mu}{d\lambda} = \frac{d\tau}{d\lambda}\frac{d\tilde{x}^\mu}{d\tau}$$

Then the equation for $d\tau/d\lambda$ is $\dfrac{d\tau}{d\lambda} = \dfrac{1}{c}\dfrac{d\tau}{d\lambda}\sqrt{\sum g_{\mu\nu}\dfrac{d\tilde{x}^\mu}{d\tau}\dfrac{d\tilde{x}^\nu}{d\tau}}$. which tells us $\sum g_{\mu\nu}\dfrac{d\tilde{x}^\mu}{d\tau}\dfrac{d\tilde{x}^\nu}{d\tau} = c^2$. Plugging the chain rule into (1) gives

$$\begin{aligned}\frac{d\tau}{d\lambda}\frac{d}{d\tau}&\left(mc\frac{\sum_\nu g_{\mu\nu}(\tilde{x})d\tilde{x}^\nu/d\tau}{\sqrt{\sum_{\alpha\beta} g_{\alpha\beta}(\tilde{x})\,d\tilde{x}^\alpha/d\tau\,d\tilde{x}^\beta/d\tau}}\right)\\ &= \frac{d\tau}{d\lambda}mc\sum_{\alpha\beta}\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial\tilde{x}^\mu}\frac{d\tilde{x}^\alpha/d\tau\,d\tilde{x}^\beta/d\tau}{\sqrt{\sum g_{\mu\nu}\,d\tilde{x}^\mu/d\tau\,d\tilde{x}^\nu/d\tau}}.\end{aligned}$$

Assuming $d\tau/d\lambda$ is nonsingular, we may cancel it from both sides, and as we now know the square roots now have arguments c^2, this reduces to

$$\frac{d}{d\tau}\left(\sum_\nu g_{\mu\nu}(\tilde{x})\frac{d\tilde{x}^\nu}{d\tau}\right) = \sum_{\alpha\beta}\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial\tilde{x}^\mu}\frac{d\tilde{x}^\alpha}{d\tau}\frac{d\tilde{x}^\beta}{d\tau}.$$

If we then define $p_\mu = m\sum_\nu g_{\mu\nu}(\tilde{x})\dfrac{d\tilde{x}^\nu}{d\tau}$, we have extablished what we needed for our solution.