Physics 507 Homework #10
Due: Thursday, Nov. 18, 2010

10.1 [20 points] We have considered k-forms in 3-D Euclidean space and their relation to vectors expressed in cartesian basis vectors. We have seen that k-forms are invariant under change of coordinatization of $\mathcal{M}$, so we can use them to examine the forms of the gradient, curl, divergence and laplacian in general coordinates in three dimensional space. We will restrict our treatment to *orthogonal curvilinear coordinates* (q_1, q_2, q_3), for which we have, at each point $\mathbf{p} \in \mathcal{M}$, a set of **orthonormal** basis vectors $\hat{e}_i$ directed along the corresponding coordinate, so that $dq_i(\hat{e}_j) = 0$ for $i \neq j$. We assume they are right handed, so $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ and $\hat{e}_i \times \hat{e}_j = \sum_k \epsilon_{ijk}\hat{e}_k$. The dq_i are not normalized measures of distance, so we define $h_i(\mathbf{p})$ so that $dq_i(\hat{e}_j) = h_i^{-1}\delta_{ij}$ (no sum).

(a) For a function $f(q_1, q_2, q_3)$ and a vector $\vec{v} = \sum v_i\hat{e}_i$, we know that $df(\vec{v}) = \vec{v} \cdot \vec{\nabla} f$. Use this to find the expression for $\vec{\nabla} f$ in the basis $\hat{e}_i$.
(b) Use this to get the general relation of a 1-form $\sum \omega_i dq_i$ to its associated vector $\vec{v} = \sum v_i\hat{e}_i$.
(c) If a 1-form $\omega^{(a)}$ is associated with $\vec{v}^{(a)}$ and 1-form $\omega^{(b)}$ is associated with $\vec{v}^{(b)}$, we know the 2-form $\omega^{(a)} \wedge \omega^{(b)}$ is associated with $\vec{v}^{(a)} \times \vec{v}^{(b)}$. Use this to find the general association of a 2-form with a vector.
(d) We know that if a 1-form ω is associated with a vector $vecv$, then $d\omega$ is associated with $\vec{\nabla}\times\vec{v}$. Use this to find the expression for $\vec{\nabla}\times\vec{v}$ in orthogonal curvilinear coordinates.
(e) If the 1-form ω is associated with $\vec{v}$ and the 2-form Ω is associated with $\vec{F}$, we know that $\omega \wedge \Omega$ is associated with the scalar $\vec{v} \cdot \vec{F}$. Use this to find the general association of a 3-form with a scalar.
(f) If the 2-form Ω is associated with $\vec{v}$, we know that $d\Omega$ is associated with the divergence of $\vec{v}$. Use this to find the expression for $\vec{\nabla} \cdot \vec{v}$ in orthogonal curvilinear coordinates.
(g) Use (a) and (f) to find the expression for the laplacian of a scalar, $\nabla^2 f = \vec{\nabla} \cdot \vec{\nabla} f$, in orthogonal curvilinear coordinates.

10.2 Consider the unusual Hamiltonian for a one-dimensional problem

$$H = \omega(x^2 + 1)p,$$

where ω is a constant.

(a) Find the equations of motion, and solve for $x(t)$.

(b) Consider the transformation to new phase-space variables $P = \alpha p^{\frac{1}{2}}$, $Q = \beta x p^{\frac{1}{2}}$. Find the conditions necessary for this to be a canonical transformation, and find a generating function $F(x, Q)$ for this transformation.

(c) What is the Hamiltonian in the new coordinates?