10.1 [20 points] We have considered \( k \)-forms in 3-D Euclidean space and their relation to vectors expressed in cartesian basis vectors. We have seen that \( k \)-forms are invariant under change of coordinatization of \( M \), so we can use them to examine the forms of the gradient, curl, divergence and laplacian in general coordinates in three dimensional space. We will restrict our treatment to orthogonal curvilinear coordinates \((q_1, q_2, q_3)\), for which we have, at each point \( p \in M \), a set of orthonormal basis vectors \( \hat{e}_i \) directed along the corresponding coordinate, so that \( dq_i(\hat{e}_j) = 0 \) for \( i \neq j \). We assume they are right handed, so \( \hat{e}_i \cdot \hat{e}_j = \delta_{ij} \) and \( \hat{e}_i \times \hat{e}_j = \sum_k \epsilon_{ijk} \hat{e}_k \). The \( dq_i \) are not normalized measures of distance, so we define \( h_i(p) \) so that \( dq_i(\hat{e}_j) = h_i^{-1} \delta_{ij} \) (no sum).

(a) For a function \( f(q_1, q_2, q_3) \) and a vector \( \vec{v} = \sum v_i \hat{e}_i \), we know that \( df(\vec{v}) = \vec{v} \cdot \vec{\nabla} f \). Use this to find the expression for \( \vec{\nabla} f \) in the basis \( \hat{e}_i \).

(b) Use this to get the general relation of a 1-form \( \sum \omega_i dq_i \) to its associated vector \( \vec{v} = \sum v_i \hat{e}_i \).

(c) If a 1-form \( \omega^{(a)} \) is associated with \( \vec{\nu}^{(a)} \) and 1-form \( \omega^{(b)} \) is associated with \( \vec{\nu}^{(b)} \), we know the 2-form \( \omega^{(a)} \wedge \omega^{(b)} \) is associated with \( \vec{\nu}^{(a)} \times \vec{\nu}^{(b)} \). Use this to find the general association of a 2-form with a vector.

(d) We know that if a 1-form \( \omega \) is associated with a vector \( \text{vec} \vec{v} \), then \( d\omega \) is associated with \( \vec{\nabla} \times \vec{v} \). Use this to find the expression for \( \vec{\nabla} \times \vec{v} \) in orthogonal curvilinear coordinates.

(e) If the 1-form \( \omega \) is associated with \( \vec{v} \) and the 2-form \( \Omega \) is associated with \( \vec{F} \), we know that \( \omega \wedge \Omega \) is associated with the scalar \( \vec{v} \cdot \vec{F} \). Use this to find the general association of a 3-form with a scalar.

(f) If the 2-form \( \Omega \) is associated with \( \vec{v} \), we know that \( d\Omega \) is associated with the divergence of \( \vec{v} \). Use this to find the expression for \( \vec{\nabla} \cdot \vec{v} \) in orthogonal curvilinear coordinates.

(g) Use (a) and (f) to find the expression for the laplacian of a scalar, \( \nabla^2 f = \vec{\nabla} \cdot \vec{\nabla} f \), in orthogonal curvilinear coordinates.

10.2 Consider the unusual Hamiltonian for a one-dimensional problem

\[
H = \omega(x^2 + 1)p,
\]

where \( \omega \) is a constant.

(a) Find the equations of motion, and solve for \( x(t) \).