Chapter 7

Perturbation Theory

The class of problems in classical mechanics which are amenable to exact solution is quite limited, but many interesting physical problems differ from such a solvable problem by corrections which may be considered small. One example is planetary motion, which can be treated as a perturbation on a problem in which the planets do not interact with each other, and the forces with the Sun are purely Newtonian forces between point particles. Another example occurs if we wish to find the first corrections to the linear small oscillation approximation to motion about a stable equilibrium point. The best starting point is an **integrable system**, for which we can find sufficient integrals of the motion to give the problem a simple solution in terms of action-angle variables as the canonical coordinates on phase space. One then phrases the full problem in such a way that the perturbations due to the extra interactions beyond the integrable forces are kept as small as possible. We first examine the solvable starting point.

7.1 Integrable systems

An **integral of the motion** for a hamiltonian system is a function F on phase space $\mathcal{M}$ for which the Poisson bracket with H vanishes, $[F, H] = 0$. More generally, a set of functions on phase space is said to be in **involution** if all their pairwise Poisson brackets vanish. The systems we shall consider are **integrable systems** in the sense that there exists one integral of the motion for each degree of freedom, and these are in involution and independent. Thus on the $2n$-dimensional manifold of phase space, there are n functions

F_i for which $[F_i, F_j] = 0$, and the F_i are independent, so the dF_i are linearly independent at each point $\eta \in \mathcal{M}$. We will assume the first of these is the Hamiltonian. As each of the F_i is a conserved quantity, the motion of the system is confined to a submanifold of phase space determined by the initial values of these invariants $f_i = F_i(q(0), p(0))$:

$$\mathcal{M}_{\vec{f}} = \{\eta : F_i(\eta) = f_i \text{ for } i = 1, \dots, n, \quad \text{connected}\},$$

where if the space defined by $F_i(\eta) = f_i$ is disconnnected, $\mathcal{M}_{\vec{f}}$ is only the piece in which the system starts. The differential operators $D_{F_i} = [F_i, \cdot]$ correspond to vectors tangent to the manifold $\mathcal{M}_{\vec{f}}$, because acting on each of the F_j functions, D_{F_i} vanishes, as the F's are in involution. These differential operators also commute with one another, because as we saw in (6.13),

$$D_{F_i} D_{F_j} - D_{F_j} D_{F_i} = D_{[F_i, F_j]} = 0.$$

They are also linearly independent, for if $\sum \alpha_i D_{F_i} = 0$, $\sum \alpha_i D_{F_i} \eta_j = 0 = [\sum \alpha_i F_i, \eta_j]$, which means that $\sum \alpha_i F_i$ is a constant on all of phase space, and that would contradict the assumed independence of the F_i. Thus the D_{F_i} are n commuting independent differential operators corresponding to the generators F_i of an Abelian[1] group of displacements on $\mathcal{M}_{\vec{f}}$. A given reference point $\eta_0 \in \mathcal{M}$ is mapped by the canonical transformation generator $\sum t_i F_i$ into some other point $g^{\vec{t}}(\eta_0) \in \mathcal{M}_{\vec{f}}$. Poisson's Theorem shows the volume covered diverges with $\vec{t}$, so if the manifold $\mathcal{M}_{\vec{f}}$ is compact, there must be many values of $\vec{t}$ for which $g^{\vec{t}}(\eta_0) = \eta_0$. These elements form a discrete Abelian subgroup, and therefore a lattice in $\mathbb{R}^n$. It has n independent lattice vectors, and a unit cell which is in 1-1 correspondence with $\mathcal{M}_{\vec{f}}$. Let these basis vectors be $\vec{e}_1, \dots, \vec{e}_n$. These are the edges of the unit cell in $\mathbb{R}^n$, the interior of which is a linear combination $\sum a_i \vec{e}_i$ where each of the $a_i \in [0, 1)$. We therefore have a diffeomorphism between this unit cell and $\mathcal{M}_{\vec{f}}$, which induces coordinates on $\mathcal{M}_{\vec{f}}$. Because these are periodic, we scale the a_i to new coordinates $\phi_i = 2\pi a_i$, so each point of $\mathcal{M}_{\vec{f}}$ is labelled by $\vec{\phi}$, given by the $\vec{t} = \sum \phi_k \vec{e}_k / 2\pi$ for which $g^{\vec{t}}(\eta_0) = \eta$. Notice each ϕ_i is a coordinate on a circle, with $\phi_i = 0$ representing the same point as $\phi_i = 2\pi$, so the manifold $\mathcal{M}_{\vec{f}}$ is diffeomorphic to an n dimensional torus $T^n = (S^1)^n$.

[1] An Abelian group is one whose elements all commute with each other, $A \odot B = B \odot A$ for all $A, B \in G$. When Abelian group elements are expressed as exponentials of a set of independent infinitesimal generators, group multiplication corresponds to addition of the parameters multiplying the generators in the exponent.

Under an infinitesimal generator $\sum \delta t_i F_i$, a point of $\mathcal{M}_{\vec{f}}$ is translated by $\delta\eta = \sum \delta t_i [\eta, F_i]$. This is true for any choice of the coordinates η, in particular it can be applied to the ϕ_j, so

$$\delta\phi_j = \sum_i \delta t_i [\phi_j, F_i],$$

where we have already expressed

$$\delta\vec{t} = \sum_k \delta\phi_k \vec{e}_k / 2\pi.$$

We see that the Poisson bracket is the inverse of the matrix A_{ji} given by the j'th coordinate of the i'th basis vector

$$A_{ji} = \frac{1}{2\pi} \left(\vec{e}_i\right)_j, \qquad \delta\vec{t} = A \cdot \delta\phi, \qquad [\phi_j, F_i] = \left(A^{-1}\right)_{ji}.$$

As the Hamiltonian $H = F_1$ corresponds to the generator with $\vec{t} = (1, 0, \dots, 0)$, an infinitesimal time translation generated by $\delta t H$ produces a change $\delta\phi_i = (A^{-1})_{i1}\delta t = \omega_i \delta t$, for some vector $\vec{\omega}$ which is determined by the $\vec{e}_i$. Note that the periodicities $\vec{e}_i$ may depend on the values of the integrals of the motion, so $\vec{\omega}$ does as well, and we have

$$\frac{d\vec{\phi}}{dt} = \vec{\omega}(\vec{f}).$$

The angle variables $\vec{\phi}$ are not conjugate to the integrals of the motion F_i, but rather to combinations of them,

$$I_i = \frac{1}{2\pi} \vec{e}_i(\vec{f}) \cdot \vec{F},$$

for then

$$[\phi_j, I_i] = \frac{1}{2\pi} \left(\vec{e}_i(\vec{f})\right)_k [\phi_j, F_k] = A_{ki} \left(A^{-1}\right)_{jk} = \delta_{ij}.$$

These I_i are the action variables, which are functions of the original set F_j of integrals of the motion, and therefore are themselves integrals of the motion. In action-angle variables the motion is very simple, with $\vec{I}$ constant and $\dot{\vec{\phi}} = \vec{\omega} = \text{constant}$. This is called **conditionally periodic motion**, and the ω_i are called the frequencies. If all the ratios of the ω_i's are rational, the motion will be truly periodic, with a period the least common multiple of the individual periods $2\pi/\omega_i$. More generally, there may be some relations

$$\sum_i k_i \omega_i = 0$$

for *integer* values k_i. Each of these is called a **relation among the frequencies**. If there are no such relations the frequencies are said to be **independent frequencies**.

In the space of possible values of $\vec{\omega}$, the subspace of values for which the frequencies are independent is surely dense. In fact, most such points have independent frequencies. We should be able to say then that most of the invariant tori $\mathcal{M}_{\vec{f}}$ have independent frequencies if the mapping $\vec{\omega}(\vec{f})$ is one-to-one. This condition is

$$\det\left(\frac{\partial\vec{\omega}}{\partial\vec{f}}\right) \neq 0, \quad \text{or equivalently} \quad \det\left(\frac{\partial\vec{\omega}}{\partial\vec{I}}\right) \neq 0.$$

When this condition holds the system is called a **nondegenerate system**. As $\omega_i = \partial H/\partial I_i$, this condition can also be written as $\det \partial^2 H/\partial_{I_i}\partial_{I_j} \neq 0$.

Consider a function g on $\mathcal{M}_{\vec{f}}$. We define two averages of this function. One is the time average we get starting at a particular point $\vec{\phi}_0$ and averaging over over an infinitely long time,

$$\langle g\rangle_t(\vec{\phi}_0) = \lim_{T\to\infty}\frac{1}{T}\int_0^T g(\vec{\phi}_0 + \vec{\omega}t)dt.$$

We may also define the average over phase space, that is, over all values of $\vec{\phi}$ describing the submanifold $\mathcal{M}_{\vec{f}}$,

$$\langle g\rangle_{\mathcal{M}_{\vec{f}}} = (2\pi)^{-n}\int_0^{2\pi}\dots\int_0^{2\pi} g(\vec{\phi})d\phi_1\dots d\phi_n,$$

where we have used the simple measure $d\phi_1\dots d\phi_n$ on the space $\mathcal{M}_{\vec{f}}$. Then an important theorem states that, if the frequencies are independent, and g is a continuous function on $\mathcal{M}_{\vec{f}}$, the time and space averages of g are the same. Note any such function g can be expanded in a Fourier series, $g(\vec{\phi}) = \sum_{\vec{k}\in\mathbb{Z}^n} g_{\vec{k}} e^{i\vec{k}\cdot\vec{\phi}}$, with $\langle g\rangle_{\mathcal{M}_{\vec{f}}} = g_{\vec{0}}$, while

$$\begin{aligned}\langle g\rangle_t &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{\vec{k}} g_{\vec{k}}\; e^{i\vec{k}\cdot\vec{\phi}_0 + i\vec{k}\cdot\vec{\omega}t}dt \\ &= g_{\vec{0}} + \sum_{\vec{k}\neq\vec{0}} g_{\vec{k}}\; e^{i\vec{k}\cdot\vec{\phi}_0}\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i\vec{k}\cdot\vec{\omega}t}dt = g_{\vec{0}},\end{aligned}$$

because

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i\vec{k}\cdot\vec{\omega}t} = \lim_{T\to\infty}\frac{1}{T}\frac{e^{i\vec{k}\cdot\vec{\omega}T}-1}{i\vec{k}\cdot\vec{\omega}} = 0,$$

as long as the denominator does not vanish. It is this requirement that $\vec{k}\cdot\vec{\omega}\neq 0$ for all nonzero $\vec{k}\in\mathbb{Z}^n$, which requires the frequencies to be independent.

As an important corrolary of this theorem, when it holds the trajectory is dense in $\mathcal{M}_{\vec{f}}$, and uniformly distributed, in the sense that the time spent in each specified volume of $\mathcal{M}_{\vec{f}}$ is proportional to that volume, independent of the position or shape of that volume. This leads to the notion of **ergodicity**, that every state of a system left for a long time will have average values of various properties the same as the average of all possible states with the same conserved values.

If instead of independence we have relations among the frequencies, these relations, each given by a $\vec{k}\in\mathbb{Z}^n$, form a subgroup of $\mathbb{Z}^n$ (an additive group of translations by integers along each of the axes). Each such $\vec{k}$ gives a constant of the motion, $\vec{k}\cdot\vec{\phi}$. Each independent relation among the frequencies therefore restricts the dimensionality of the motion by an additional dimension, so if the subgroup is generated by r such independent relations, the motion is restricted to a manifold of reduced dimension $n-r$, and the motion on this reduced torus T^{n-r} is conditionally periodic with $n-r$ independent frequencies. The theorem and corrolaries just discussed then apply to this reduced invariant torus, but not to the whole n-dimensional torus with which we started. In particular, $\langle g\rangle_t(\phi_0)$ can depend on ϕ_0 as it varies from one submanifold T^{n-r} to another, but not along paths on the same submanifold.

While having relations among the frequencies for arbitrary values of the integrals of the motion might seem a special case, unlikely to happen, there are important examples where they do occur. We saw that for Keplerian motion, there were five invariant functions on the six-dimensional phase space of the relative coordinate, because energy, angular momentum, and the Runge-Lenz are all conserved, giving five independent conserved quantities. The locus of points in the six dimensional space with these five functions taking on assigned values is therefore one-dimensional, that is, a curve on the three dimensional invariant torus. This is responsible for the stange fact that the oscillations in r have the same period as the cycles in ϕ. Even for other central force laws, for which there is no equivalent to the Runge-Lenz vector, there are still four conserved quantities, so there must still be one relation, which turns out to be that the periods of motion in θ and ϕ are the same[2].

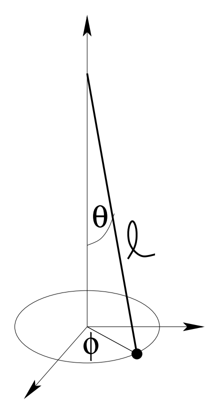
If the system is nondegenerate, for typical $\vec{I}$ the ω_i's will have no relations and the invariant torus will be densely filled by the motion of the system. Therefore the invariant tori are uniquely defined, although the choices of action and angle variables is not. In the degenerate case the motion of the system does not fill the n dimensional invariant torus, so it need not be uniquely defined. This is what happens, for example, for the two dimensional harmonic oscillator or for the Kepler problem.

This discussion has been somewhat abstract, so it might be well to give some examples. We will consider

- the pendulum
- the two-dimensional isotropic harmonic oscillator
- the three dimensional isotropic anharmonic oscillator

The Pendulum

The simple pendulum is a mass connected by a fixed length massless rod to a frictionless ball joint, which we take to be at the origin, hanging in a uniform gravitational field. The generalized coordinates may be taken to be the angle θ which the rod makes with the **downward** vertical, and the azimuthal angle ϕ. If ℓ is the length of the rod, $U = -mg\ell\cos\theta$, and as shown in section 2.2.1 or section 3.1.2, the kinetic energy is $T = \frac{1}{2}m\ell^2\left(\dot\theta^2 + \sin^2\theta\dot\phi^2\right)$. So the lagrangian,



$$L = \frac{1}{2}m\ell^2\left(\dot\theta^2 + \sin^2\theta\dot\phi^2\right) + mg\ell\cos\theta$$

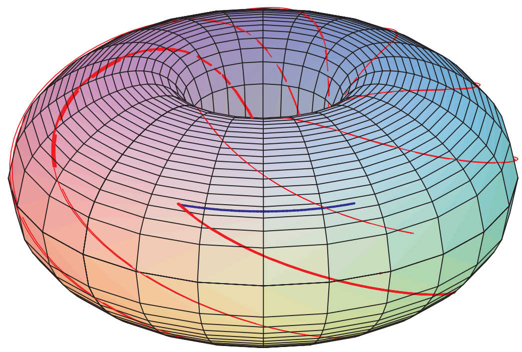
is time independent and has an ignorable coordinate ϕ,

[2]The usual treatment for spherical symmetry is to choose $\vec{L}$ in the z direction, which sets z and p_z to zero and reduces our problem to a four-dimensional phase space with two integrals of the motion, H and L_z. But without making that choice, we do know that the motion will be resticted to some plane, so $a_xx + a_yy + a_zz = 0$ for some fixed coefficients a_x, a_y, a_z, and in spherical coordinates $r(a_z\cos\theta + a_x\sin\theta\cos\phi + a_y\sin\theta\sin\phi) = 0$. The r dependence factors out, and thus ϕ can be solved for, in terms of θ, and must have the same period.

so $p_\phi = m\ell^2 \sin^2\theta\dot\phi$ is conserved, and so is H. As $p_\theta = m\ell^2\dot\theta$, the Hamiltonian is

$$H = \frac{1}{2m\ell^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - mg\ell\cos\theta.$$

In the four-dimensional phase space one coordinate, p_ϕ, is fixed, and the equation $H(\theta, \phi, p_\theta) = E$ gives a two-dimensional surface in the three-dimensional space which remains. Let us draw this in cylindrical coordiates with radial coordinate θ, angular coordinate ϕ, and z coordinate p_θ, and polar angle ϕ. Thus the motion will be restricted to the **invariant torus** shown. The generators $F_2 = p_\phi$ and $F_1 = H$ generate motions along the torus as shown, with p_ϕ generating changes in ϕ, leaving θ and p_θ fixed. Thus a point moves as on the blue path shown, looking like a line of latitude. The change in ϕ generated by $g^{(0,t_2)}$ is just t_2, so we may take $\phi = \phi_2$ of the last section. H generates the dynamical motion of the system,



$$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2}, \qquad \dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\ell^2\sin^2\theta},$$

$$\dot p_\theta = -\frac{\partial H}{\partial\theta} = \frac{p_\phi^2\cos\theta}{m\ell^2\sin^3\theta} - mg\ell\sin\theta.$$

This is shown by the red path, which goes around the bottom, through the hole in the donut, up the top, and back, but not quite to the same point as it started. Ignoring ϕ, this is periodic motion in θ with a period T_θ, so $g^{(T_\theta,0)}(\eta_0)$ is a point at the same latitude as η_0. This $t \in [0, T_\theta]$ part of the trajectory is shown as the thick red curve. There is some $\bar t_2$ which, together with $\bar t_1 = T_\theta$, will cause $g^{\vec{\bar t}}$ to map each point on the torus back to itself.

Thus $\vec e_1 = (T_\theta, \bar t_2)$ and $\vec e_2 = (0, 2\pi)$ constitute the unit vectors of the lattice of $\vec t$ values which leave the points unchanged. The trajectory generated by H does not close after one or a few T_θ. It could be continued indefinitely, and as in general there is no relation among the frequencies ($\bar t_2/2\pi$ is not rational, in general), the trajectory will not close, but will fill the surface of the torus. If we wait long enough, the system will sample every region of the torus.

The 2-D isotropic harmonic oscillator

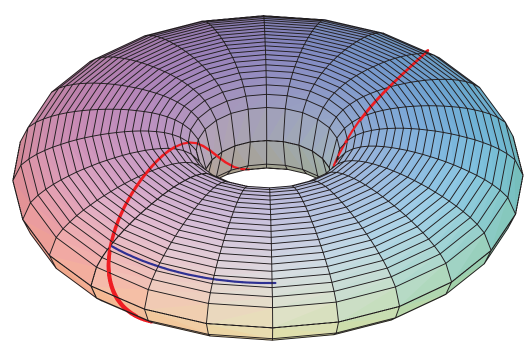
A different result occurs for the two dimensional zero-length isotropic oscillator,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}kr^2.$$

While this separates in cartesian coordinates, from which we easily see that the orbit closes because the two periods are the same, we will look instead at polar coordinates, where we have a conserved Hamiltonian

$$F_1 = H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + \frac{1}{2}kr^2,$$

and conserved momentum p_ϕ conjugate to the ignorable coordinate ϕ. As before, p_ϕ simply changes ϕ, as shown in blue. But now if we trace the action of H,



$$\frac{dr}{dt} = p_r(t)/m, \quad \frac{d\phi}{dt} = \frac{p_\phi}{mr^2},$$

$$\frac{dp_r}{dt} = \frac{p_\phi^2}{mr^3(t)} - kr(t),$$

we get the red curve which closes on itself after one revolution in ϕ and two trips through the donut hole. Thus the orbit is a closed curve, there is a relation among the frequencies. Of course the system now only samples the points on the closed curve, so a time average of any function on the trajectory is not the same as the average over the invariant torus.

The 3-D isotropic anharmonic oscillator

Now consider the spherically symmetric oscillator for which the potential energy is not purely harmonic, say $U(r) = \frac{1}{2}kr^2 + cr^4$. Then the Hamiltonian in spherical coordinates is

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + \frac{1}{2}kr^2 + cr^4.$$

This is time independent, so $F_1 = H$ is conserved, the first of our integrals of the motion. Also ϕ is an ignorable coordinate, so $F_2 = p_\phi = L_z$ is the second. But we know that all of $\vec{L}$ is conserved. While L_x is an integral of the motion, it is not in involution with L_z, as $[L_z, L_x] = L_y \neq 0$, so it will not serve as an additional generator. But $L^2 = \sum_k L_k^2$ is also conserved and has zero Poisson bracket with H and L_z, so we can take it to be the third generator

$$\begin{aligned} F_3 = L^2 &= (\vec{r} \times \vec{p})^2 = r^2 \vec{p}^2 - (\vec{r} \cdot \vec{p})^2 = r^2 \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) - r^2 p_r^2 \\ &= p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta}. \end{aligned}$$

The full phase space is six dimensional, and as p_ϕ is constant we are left, in general, with a five dimensional space with two nonlinear constraints. On the three-dimensional hypersurface, p_ϕ generates motion only in ϕ, the Hamiltonian generates the dynamical trajectory with changes in r, p_r, θ, p_θ and ϕ, and F_3 generates motion in θ, p_θ and ϕ, but not in r or p_r.

Now while L_x is not in involution with the three F_i already chosen, it is a constant of the (dynamical) motion, as $[L_x, H] = 0$. But under the flow generated by $F_2 = L_z$, which generates changes in η_j proportional to $[\eta_j, L_z]$, we have

$$\begin{aligned} \frac{d}{d\lambda} L_x(g^{\lambda L_z} \vec{\eta}) &= \sum_j \frac{\partial L_x(\eta)}{\partial \eta_j} [\eta_j, L_z] = \sum_{jk} \frac{\partial L_x(\eta)}{\partial \eta_j} J_{jk} \frac{\partial L_z}{\eta_k} \\ &= [L_x, L_z] \neq 0. \end{aligned}$$

Thus the constraint on the dynamical motion that L_x is conserved tells us that motion on the invariant torus generated by L_z is inconsistent with the dynamical evolution — that the trajectory lies in a discrete subspace (two dimensional) rather than being dense in the three-dimensional invariant torus. This also shows that there must be one relation among the frequencies.

Of course we could have reached this conclusion much more easily, as we did in the last footnote, by choosing the z-axis of the spherical coordinates along whatever direction $\vec{L}$ points, so the motion restricts $\vec{r}$ to the xy plane, and throwing in p_r gives us a two-dimensional torus on which the motion remains.

7.2 Canonical Perturbation Theory

We now consider a problem with a conserved Hamiltonian which is in some sense approximated by an integrable system with n degrees of freedom. This integrable system is described with a Hamiltonian $H^{(0)}$, and we assume we have described it in terms of its action variables $I_i^{(0)}$ and angle variables $\phi_i^{(0)}$. This system is called the **unperturbed system**, and the Hamiltonian is, of course, independent of the angle variables, $H^{(0)}\left(\vec{I}^{(0)},\vec{\phi}^{(0)}\right) = H^{(0)}\left(\vec{I}^{(0)}\right)$.

The action-angle variables of the unperturbed system are a canonical set of variables for the phase space, which is still the same phase space for the full system. We write the Hamiltonian of the full system as

$$H\left(\vec{I}^{(0)},\vec{\phi}^{(0)}\right) = H^{(0)}\left(\vec{I}^{(0)}\right) + \epsilon H'\left(\vec{I}^{(0)},\vec{\phi}^{(0)}\right). \tag{7.1}$$

We have included the parameter ϵ so that we may regard the terms in H' as fixed in strength relative to each other, and still consider a series expansion in ϵ, which gives an overall scale to the smallness of the perturbation.

We might imagine that if the perturbation is small, there are some new action-angle variables I_i and ϕ_i for the full system, which differ by order ϵ from the unperturbed coordinates. These are new canonical coordinates, and may be generated by a generating function (of type 2),

$$F\left(\vec{I},\vec{\phi}^{(0)}\right) = \sum \phi_i^{(0)} I_i + \epsilon F'\left(\vec{I},\vec{\phi}^{(0)}\right) + ...$$

This is a time-independent canonical transformation, so the full Hamiltonian is the same function on phase-space whether the unperturbed or full action-angle variables are used, but has a different functional form,

$$\tilde{H}(\vec{I},\vec{\phi}) = H\left(\vec{I}^{(0)},\vec{\phi}^{(0)}\right). \tag{7.2}$$

Note that the phase space itself is described periodically by the coordinates $\vec{\phi}^{(0)}$, so the Hamiltonian perturbation H' and the generating function F' are periodic functions (with period 2π) in these variables. Thus we can expand them in Fourier series:

$$H'\left(\vec{I}^{(0)},\vec{\phi}^{(0)}\right) = \sum_{\vec{k}} H'_{\vec{k}}\left(\vec{I}^{(0)}\right) e^{i\vec{k}\cdot\vec{\phi}^{(0)}}, \tag{7.3}$$

$$F'\left(\vec{I},\vec{\phi}^{(0)}\right) = \sum_{\vec{k}} F'_{\vec{k}}\left(\vec{I}\right) e^{i\vec{k}\cdot\vec{\phi}^{(0)}}, \tag{7.4}$$

where the sum is over all n-tuples of integers $\vec{k} \in \mathbb{Z}^n$. The zeros of the new angles are arbitrary for each $\vec{I}$, so we may choose $F'_{\vec{0}}(I) = 0$.

The unperturbed action variables, on which H_0 depends, are the old momenta given by $I_i^{(0)} = \partial F/\partial \phi_i^{(0)} = I_i + \epsilon \partial F'/\partial \phi_i^{(0)} + ...$, so to first order

$$\begin{aligned} H_0\left(\vec{I}^{(0)}\right) &= H_0\left(\vec{I}\right) + \epsilon \sum_j \frac{\partial H_0}{\partial I_j^{(0)}} \frac{\partial F'}{\partial \phi_j^{(0)}} + ... \\ &= H_0\left(\vec{I}\right) + \epsilon \sum_j \omega_j^{(0)} \sum_{\vec{k}} i k_j F'_{\vec{k}}(\vec{I}) e^{i\vec{k}\cdot\vec{\phi}^{(0)}} + ..., \end{aligned} \tag{7.5}$$

where we have noted that $\partial H_0/\partial I_j^{(0)} = \omega_j^{(0)}$, the frequencies of the unperturbed problem. Thus

$$\begin{aligned} \tilde{H}\left(\vec{I},\vec{\phi}\right) &= H\left(\vec{I}^{(0)},\vec{\phi}^{(0)}\right) = H^{(0)}\left(\vec{I}^{(0)}\right) + \epsilon \sum_{\vec{k}} H'_{\vec{k}}\left(\vec{I}^{(0)}\right) e^{i\vec{k}\cdot\vec{\phi}^{(0)}} \\ &= H_0\left(\vec{I}\right) + \epsilon \sum_{\vec{k}} \left(\sum_j i k_j \omega_j^{(0)} F'_{\vec{k}}\left(\vec{I}\right) + H'_{\vec{k}}\left(\vec{I}^{(0)}\right)\right) e^{i\vec{k}\cdot\vec{\phi}^{(0)}}. \end{aligned}$$

The $\vec{I}$ are the action variables of the full Hamiltonian, so $\tilde{H}(\vec{I},\vec{\phi})$ is in fact independent of $\vec{\phi}$. In the sum over Fourier modes on the right hand side, the $\phi^{(0)}$ dependence of the terms in parentheses due to the difference of $\vec{I}^{(0)}$ from $\vec{I}$ is higher order in ϵ, so the the coefficients of $e^{i\vec{k}\cdot\vec{\phi}^{(0)}}$ may be considered constants in $\phi^{(0)}$ and therefore must vanish for $\vec{k} \neq \vec{0}$. Thus the generating function is given in terms of the Hamiltonian perturbation

$$F'_{\vec{k}} = i \frac{H'_{\vec{k}}}{\vec{k}\cdot\vec{\omega}^{(0)}\left(\vec{I}\right)}, \qquad \vec{k} \neq \vec{0}. \tag{7.6}$$

We see that there may well be a problem in finding new action variables if there is a relation among the frequencies. If the unperturbed system is not degenerate, "most" invariant tori will have no relation among the frequencies. For these values, the extension of the procedure we have described to a full power series expansion in ϵ may be able to generate new action-angle variables, showing that the system is still integrable. That this is true for sufficiently small perturbations and "sufficiently irrational" $\omega_J^{(0)}$ is the conclusion of the famous KAM theorem[3].

[3]See Arnold[2], pp 404-405, though he calls it Kolmogorov's Theorem, denying credit to himself and Moser, or Josè and Saletan[8], p. 477.

What happens if there is a relation among the frequencies? Consider a two degree of freedom system with $p\omega_1^{(0)} + q\omega_2^{(0)} = 0$, with p and q relatively prime. Then the Euclidean algorithm shows us there are integers m and n such that $pm + qn = 1$. Instead of our initial variables $\phi_i^{(0)} \in [0, 2\pi]$ to describe the torus, we can use the linear combinations

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} p & q \\ n & -m \end{pmatrix} \begin{pmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \end{pmatrix} = \mathbf{B} \cdot \begin{pmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \end{pmatrix}.$$

Then ψ_1 and ψ_2 are equally good choices for the angle variables of the unperturbed system, as $\psi_i \in [0, 2\pi]$ is a good coordinate system on the torus. The corresponding action variables are $I'_i = \left(B^{-1}\right)_{ji} I_j$, and the corresponding new frequencies are

$$\omega'_i = \frac{\partial H}{\partial I'_i} = \sum_j \frac{\partial H}{\partial I_j} \frac{\partial I_j}{\partial I'_i} = B_{ij}\omega_j^{(0)},$$

and so in particular $\omega'_1 = p\omega_1^{(0)} + q\omega_2^{(0)} = 0$ on the chosen invariant torus. This conclusion is also obvious from the equations of motion $\dot{\phi}_i = \omega_i$.

In the unperturbed problem, on our initial invariant torus, ψ_1 is a constant of the motion, so in the perturbed system we might expect it to vary slowly with respect to ψ_2. Then it is appropriate to use the adiabatic approximation of section 7.3

7.2.1 Time Dependent Perturbation Theory

Now we will consider problems for which the Hamiltonian H is approximately that of an exactly solvable problem, H_0. So we write

$$H(q, p, t) = H_0(q, p, t) + \epsilon H_I(q, p, t),$$

where $\epsilon H_I(q, p, t)$ is considered a small "interaction" Hamiltonian. We assume we know Hamilton's principal function $S_0(q, P, t)$ for the unperturbed problem, which gives a canonical transformation $(q, p) \to (Q, P)$, and in the limit $\epsilon \to 0$, $\dot{Q} = \dot{P} = 0$. For the full problem,

$$K(Q, P, t) = H_0 + \epsilon H_I + \frac{\partial S_0}{\partial t} = \epsilon H_I,$$

and is small. Expressing H_I in terms of the new variables (Q, P), we have that

$$\dot{Q} = \epsilon \frac{\partial H_I}{\partial P}, \qquad \dot{P} = -\epsilon \frac{\partial H_I}{\partial Q}$$

and these are slowly varying because ϵ is small. In symplectic form, with $\zeta^T = (Q, P)$, we have, of course,

$$\dot{\zeta} = \epsilon J \cdot \nabla H_I(\zeta). \tag{7.7}$$

This differential equation can be solved perturbatively. If we assume an expansion

$$\zeta(t) = \zeta_0(t) + \epsilon\zeta_1(t) + \epsilon^2\zeta_2(t) + ...,$$

$\dot{\zeta}_n$ on the left of (7.7) can be determined from only lower order terms ζ_j, $j < n$ on the right hand side. The initial value $\zeta(0)$ is arbitrary, so we can take it to be $\zeta_0(0)$, and determine $\zeta_n(t) = \int_0^t \dot{\zeta}_n(t')dt'$ accurate to order ϵ^n. Thus we can recursively find higher and higher order terms in ϵ. This is a good expansion for ϵ small enough, for fixed t, but as we are making an error in $\dot{\zeta}$, this will give an error of order ϵt compared to the previous stage., so the total error at the m'th step is $\mathcal{O}([\epsilon t]^m)$ for $\zeta(t)$. Thus for calculating the long time behavior of the motion, this method is unlikely to work in the sense that any finite order calculation cannot be expected to be good for $t \to \infty$. Even though H and H_0 differ only slightly, and so acting on any given η they will produce only slightly different rates of change, as time goes on there is nothing to prevent these differences from building up. In a periodic motion, for example, the perturbation is likely to make a change $\Delta\tau$ of order ϵ in the period τ of the motion, so at a time $t \sim \tau^2/2\Delta\tau$ later, the systems will be at opposite sides of their orbits, not close together at all.

Clearly a better approximation scheme is called for, one in which $\zeta(t)$ is compared to $\zeta_0(t')$ for a more appropriate time t'. The canonical method does this, because it compares the full Hamiltonian and the unperturbed one at given values of ϕ, not at a given time. Another example of such a method applies to adiabatic invariants.

7.3 Adiabatic Invariants

7.3.1 Introduction

We are going to discuss the evolution of a system which is, at every instant, given by an integrable Hamiltonian, but for which the parameters of that Hamiltonian are slowly varying functions of time. We will find that this leads to an approximation in which the actions are time invariant. We begin with a qualitative discussion, and then we discuss a formal perturbative expansion.

First we will consider a system with one degree of freedom described by a Hamiltonian $H(q, p, t)$ which has a slow time dependence. Let us call T_V the time scale over which the Hamiltonian has significant variation (for fixed q, p). For a short time interval $\ll T_V$, such a system could be approximated by the Hamiltonian $H_0(q, p) = H(q, p, t_0)$, where t_0 is a fixed time within that interval. Any perturbative solution based on this approximation may be good during this time interval, but if extended to times comparable to the time scale T_V over which $H(q, p, t)$ varies, the perturbative solution will break down. We wish to show, however, that if the motion is bounded and the period of the motion determined by H_0 is much less than the time scale of variations T_V, the action is very nearly conserved, even for evolution over a time interval comparable to T_V. We say that the action is an **adiabatic invariant**.

7.3.2 For a time-independent Hamiltonian

In the absence of any explicit time dependence, a Hamiltonian is conserved. The motion is restricted to lie on a particular contour $H(q, p) = \alpha$, for all times. For bound solutions to the equations of motion, the solutions are periodic closed orbits in phase space. We will call this contour Γ, and the period of the motion τ. Let us parameterize the contour with the **action-angle** variable ϕ. We take an arbitrary point on Γ to be $\phi = 0$ and also $(q(0), p(0))$. As action-angles evolve at a fixed rate, every other point is determined by $\Gamma(\phi) = (q(\phi\tau/2\pi), p(\phi\tau/2\pi))$, so the complete orbit is given by $\Gamma(\phi), \phi \in [0, 2\pi)$. The action is defined as

$$J = \frac{1}{2\pi}\oint p dq. \tag{7.8}$$

This may be considered as an integral along one cycle in extended phase space, $2\pi J(t) = \int_t^{t+\tau} p(t')\dot{q}(t')dt'$. Because $p(t)$ and $\dot{q}(t)$ are periodic with period τ, J is independent of time t. But J can also be thought of as an integral in phase space itself, $2\pi J = \oint_\Gamma p dq$, of a one form $\omega_1 = pdq$ along the closed path $\Gamma(\phi)$, $\phi \in [0, 2\pi]$, which is the orbit in question. By Stokes' Theorem,

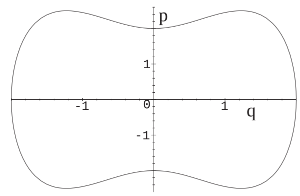


Fig. 1. The orbit of an autonomous system in phase space.

$$\int_S d\omega = \int_{\delta S} \omega,$$

true for any n-form ω and suitable region S of a manifold, we have $2\pi J = \int_A dp \wedge dq$, where A is the area bounded by Γ.

In extended phase space $\{q, p, t\}$, if we start at time t=0 with any point (q, p) on Γ, the trajectory swept out by the equations of motion, $(q(t), p(t), t)$ will lie on the surface of a cylinder with base A extended in the time direction. Let Γ_t be the embedding of Γ into the time slice at t, which is the intersection of the cylinder with that time slice. The surface of the cylinder can also be viewed as the set of all the dynamical trajectories which start anywhere on Γ at $t = 0$. In other words, if $\mathcal{T}_\phi(t)$ is the trajectory of the system which starts at $\Gamma(\phi)$ at t=0, the set of $\mathcal{T}_\phi(t)$ for $\phi \in [0, 2\pi]$, $t \in [0, T]$, sweeps out the same surface as $\{\Gamma_t\}$, for all $t \in [0, T]$. Because this is an autonomous system, the value of the action J is the same, regardless of whether it is evaluated along Γ_t, for any t, or evaluated along one period for any of the trajectories starting on Γ_0. If we terminate the evolution at time T, the end of the cylinder, Γ_T, is the same orbit of the motion, in phase space, as was Γ_0.

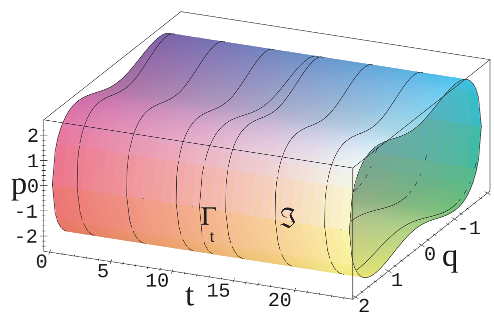


Fig 2. The surface in extended phase space, generated by the ensemble of systems which start at time $t = 0$ on the orbit Γ shown in Fig. 1. One such trajectory is shown, labelled $\mathcal{I}$, and also shown is one of the Γ_t.

7.3.3 Slow time variation in $H(q, p, t)$

Now consider a time dependent Hamiltonian $H(q, p, t)$. For a short interval of time near t_0, if we assume the time variation of H is slowly varying, the autonomous Hamiltonian $H(q, p, t_0)$ will provide an approximation, one that has conserved energy and bound orbits given by contours of that energy. Consider extended phase space, and a closed path $\Gamma_0(\phi)$ in the t=0 plane which is a contour of $H(q, p, 0)$, just as we had in the autonomous case. For each point ϕ on this path, construct the trajectory $\mathcal{T}_\phi(t)$ evolving from $\Gamma(\phi)$ under the influence of the **full** Hamiltonian $H(q, p, t)$, up until some fixed final time $t = T$. This collection of trajectories will sweep out a curved surface Σ_1 with boundary Γ_0 at t=0 and another we call Γ_T at time t=T. Because the Hamiltonian does change with time, these Γ_t, the intersections of Σ_1 with the planes at various times t, are not congruent. Let Σ_0 and Σ_T be the regions of the t=0 and t=T planes bounded by Γ_0 and Γ_T respectively, oriented so that their normals go forward in time.

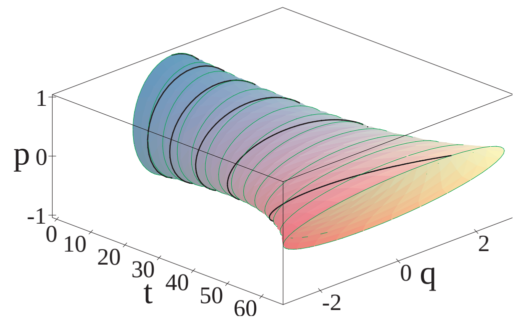


Fig. 3. The motion of a harmonic oscillator with time-varying spring constant $k \propto (1 - \epsilon t)^4$, with $\epsilon = 0.01$. [Note that the horn is not tipping downwards, but the surface ends flat against the $t = 65$ plane.]

This constructs a region which is a deformation of the cylinder[4] that we had in the case where H was independent of time. Of course if the variation of H is slow on a time scale of T, the path Γ_T will not differ much from Γ_0, so it will be nearly an orbit and the action defined by $\oint p dq$ around Γ_T will be nearly that around Γ_0. We shall show something much stronger; that if the time dependence of H is a slow variation compared with the approximate period of the motion, then each Γ_t is nearly an orbit and the action on that path, $\tilde{J}(t) = \oint_{\Gamma_t} p dq$ is constant, even if the Hamiltonian varies considerably over time T.

The Σ's form a closed surface, which is $\Sigma_1 + \Sigma_T - \Sigma_0$, where we have taken the orientation of Σ_1 to point outwards, and made up for the inward-pointing direction of Σ_0 with a negative sign. Call the volume enclosed by this closed surface V.

We will first show that the actions $\tilde{J}(0)$ and $\tilde{J}(T)$ defined on the ends of

[4]Of course it is possible that after some time, which must be on a time scale of order T_V rather than the much shorter cycle time τ, the trajectories might intersect, which would require the system to reach a critical point in phase space. We assume that our final time T is before the system reaches a critical point.

the cylinder are the same. Again from Stokes' theorem, they are

$$\tilde{J}(0) = \int_{\Gamma_0} pdq = \int_{\Sigma_0} dp \wedge dq \quad \text{and} \quad \tilde{J}(T) = \int_{\Sigma_T} dp \wedge dq$$

respectively. Each of these surfaces has no component in the t direction, so we may also evaluate $\tilde{J}(t) = \int_{\Sigma_t} d\omega_3$, where $\omega_3 = pdq - Hdt$ is the one-form of section (6.6) which determines the motion by Hamilton's principle. So

$$d\omega_3 = dp \wedge dq - dH \wedge dt. \tag{7.9}$$

Clearly $d\omega_3$ is closed as it is exact.

As H is a function on extended phase space, $dH = \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial t}dt$, and thus

$$\begin{aligned} d\omega_3 &= dp \wedge dq - \frac{\partial H}{\partial p}dp \wedge dt - \frac{\partial H}{\partial q}dq \wedge dt \\ &= \left(dp + \frac{\partial H}{\partial q}dt\right) \wedge \left(dq - \frac{\partial H}{\partial p}dt\right), \end{aligned} \tag{7.10}$$

where we have used the antisymmetry of the wedge product, $dq \wedge dt = -dt \wedge dq$, and $dt \wedge dt = 0$.

Now the interesting thing about this rewriting of the action in terms of the new form (7.10) of $d\omega_3$ is that $d\omega_3$ is now a product of two 1-forms

$$d\omega_3 = \omega_a \wedge \omega_b, \quad \text{where} \;\; \omega_a = dp + \frac{\partial H}{\partial q}dt, \quad \omega_b = dq - \frac{\partial H}{\partial p}dt,$$

and each of ω_a and ω_b vanishes along any trajectory of the motion, along which Hamilton's equations require

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}.$$

As a consequence, $d\omega_3$ vanishes at any point when evaluated on a surface which contains a physical trajectory, so in particular $d\omega_3$ vanishes over the surface Σ_1 generated by the trajectories. Because $d\omega_3$ is closed,

$$\int_{\Sigma_1+\Sigma_T-\Sigma_0} d\omega_3 = \int_V d(d\omega_3) = 0$$

where the first equality is due to Gauss' law, one form of the generalized Stokes' theorem. Then we have

$$\tilde{J}(T) = \int_{\Sigma_T} d\omega_3 = \int_{\Sigma_0} d\omega_3 = \tilde{J}(0).$$

What we have shown here for the area in phase space enclosed by an orbit holds equally well for any area in phase space. If A is a region in phase space, and if we define B as that region in phase space in which systems will lie at time $t = T$ if the system was in A at time $t = 0$, then $\int_A dp \wedge dq = \int_B dp \wedge dq$. For systems with $n > 1$ degrees of freedom, we may consider a set of n forms $(\sum_k dp_k \wedge dq_k)^j, j = 1...n$, which are all conserved under dynamical evolution. In particular, $(\sum_k dp \wedge dq_k)^n$ tells us the hypervolume in phase space is preserved under its motion under evolution according to Hamilton's equations of motion. This truth is known as Liouville's theorem, though the n invariants $(\sum_k dp \wedge dq_k)^j$ are known as Poincaré invariants.

While we have shown that the integral $\int pdq$ is conserved when evaluated over an initial contour in phase space at time $t = 0$, and then compared to its integral over the path at time $t = T$ given by the time evolution of the ensembles which started on the first path, neither of these integrals are exactly an action.

In fact, for a time-varying system the action is not really well defined, because actions are defined only for periodic motion. For the one dimensional harmonic oscillator (with varying spring constant) of Fig. 3, a reasonable substitute definition is to define J for each "period" from one passing to the right through the symmetry point, $q = 0$, to the next such crossing. The

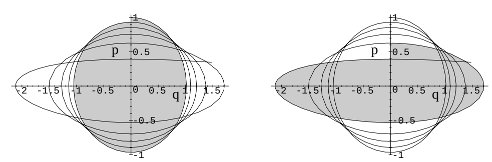


Fig. 4. The trajectory in phase space of the system in Fig. 3. The "actions" during two "orbits" are shown by shading. In the adiabatic approximation the areas are equal.

trajectory of a single such system as it moves through phase space is shown in Fig. 4. The integrals $\int p(t)dq(t)$ over time intervals between successive forward crossings of $q = 0$ is shown for the first and last such intervals. While these appear to have roughly the same area, what we have shown is that the integrals over the curves Γ_t are the same. In Fig. 5 we show Γ_t for t at the beginning of the first and fifth "periods", together with the actual motion through those periods. The deviations are of order $\epsilon\tau$ and not of ϵT, and so are negligible as long as the approximate period is small compared to $T_V \sim 1/\epsilon$.

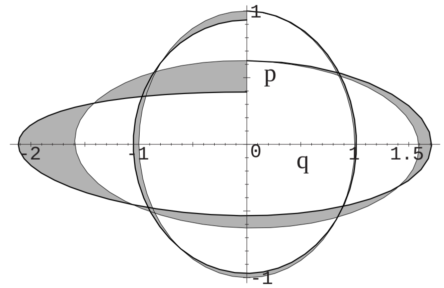


Fig. 5. The differences between the actual trajectories (thick lines) during the first and fifth oscillations, and the ensembles Γ_t at the moments of the beginnings of those periods. The area enclosed by the latter two curves are strictly equal, as we have shown. The figure indicates the differences between each of those curves and the actual trajectories.

Another way we can define an action in our time-varying problem is to write an expression for the action on extended phase space, $J(q, p, t_0)$, given by the action at that value of (q, p) for a system with hamiltonian fixed at the time in question, $H_{t_0}(q, p) := H(q, p, t_0)$. This is an ordinary harmonic oscillator with $\omega = \sqrt{k(t_0)/m}$. For an autonomous harmonic oscillator the area of the elliptical orbit is

$$2\pi J = \pi p_{\max} q_{\max} = \pi m\omega q_{\max}^2,$$

while the energy is

$$\frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 = E = \frac{m\omega^2}{2} q_{\max}^2,$$

so we can write an expression for the action as a function on extended phase space,

$$J = \frac{1}{2} m\omega q_{\max}^2 = E/\omega = \frac{p^2}{2m\omega(t)} + \frac{m\omega(t)}{2} q^2.$$

With this definition, we can assign a value for the action to the system as a each time, which in the autonomous case agrees with the standard action.

From this discussion, we see that if the Hamiltonian varies slowly on the time scale of an oscillation of the system, the action will remain fairly close to the $\tilde{J}_t$, which is conserved. Thus the action is an adiabatic invariant, conserved in the limit that $\tau/T_V \to 0$.

To see how this works in a particular example, consider the harmonic oscillator with a time-varying spring constant, which we have chosen to be $k(t) = k_0(1-\epsilon t)^4$. With $\epsilon = 0.01$, in units given by the initial ω, the evolution is shown from time 0 to time 65. During this time the spring constant becomes over 66 times weaker, and the natural frequency decreases by a factor of more than eight, as does the energy, but the action remains quite close to its original value, even though the adiabatic approximation is clearly badly violated by a spring constant which changes by a factor of more than six during the last oscillation.

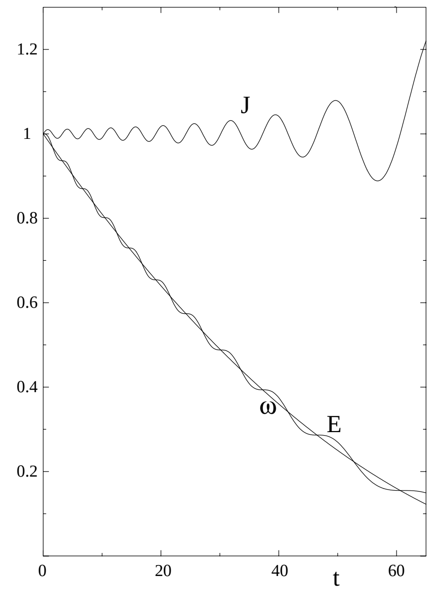


Fig. 6. The change in angular frequency, energy, and action for the time-varying spring-constant harmonic oscillator, with $k(t) \propto (1-\epsilon t)^4$, with $\epsilon = \omega(0)/100$

We see that the failure of the action to be exactly conserved is due to the descrepancy between the action evaluated on the actual path of a single system and the action evaluated on the curve representing the evolution, after a given time, of an ensemble of systems all of which began at time $t = 0$ on a path in phase space which would have been their paths had the system been autonomous.

This might tempt us to consider a different problem, in which the time dependance of the hamiltonian varies only during a fixed time interval, $t \in [0, T]$, but is constant before $t = 0$ and after T. If we look at the motion during an oscillation before $t = 0$, the system's trajectory projects exactly onto Γ_0, so the initial action $J = \tilde{J}(0)$. If we consider a full oscillation beginning after time T, the actual trajectory is again a contour of energy in phase space. Does this mean the action is exactly conserved?

There must be something wrong with this argument, because the constancy of $\tilde{J}(t)$ did not depend on assumptions of slow variation of the Hamiltonian. Thus it should apply to the pumped swing, and claim that it is impossible to increase the energy of the oscillation by periodic changes in the spring constant. But every child knows that is not correct. Examining this case will point out the flawed assumption in the argument. In Fig. 7, we show the surface generated by time evolution of an ensemble of systems initially on an energy contour for a harmonic oscillator. Starting at time 0, the spring constant is modulated by 10% at a frequency twice the natural frequency, for four natural periods. Thereafter the Hamiltonian is the same as is was before $t = 0$, and each system's path in phase space continues as a circle in phase space (in the units shown), but the ensemble of systems form a very elongated figure, rather than a circle.

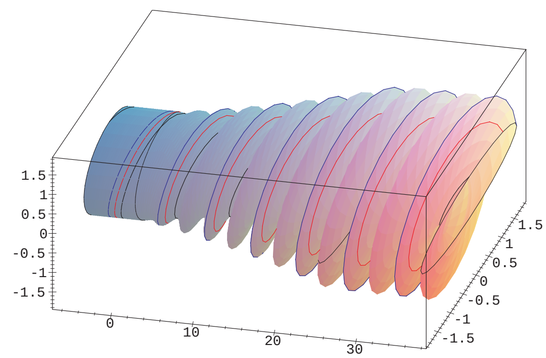


Fig. 7. The surface Σ_1 for a harmonic oscillator with a spring constant which varies, for the interval $t \in [0, 8\pi]$, as $k(t) = k(0)(1 + 0.1 \sin 2t)$.

What has happened is that some of the systems in the ensemble have gained energy from the pumping of the spring constant, while others have lost energy. Thus there has been no conservation of the action for individual systems, but rather there is some (vaguely understood) average action which is unchanged.

Thus we see what is physically the crucial point in the adiabatic expansion: if all the systems in the ensemble experience the perturbation in the same way, because the time variation of the hamiltonian is slow compared to the time it takes for each system in the ensemble to occupy the initial position (in phase space) of every other system, then each system will have its action conserved.

7.3.4 Systems with Many Degrees of Freedom

In the discussion above we considered as our starting point an autonomous system with one degree of freedom. As the hamiltonian is a conserved function on phase space, this is an integrable system. For systems with $n > 1$ degrees of freedom, we wish to again start with an integrable system. Such systems have n invariant "integrals of the motion in involution", and their phase space can be described in terms of n action variables J_i and corresponding coordinates ϕ_i. Phase space is periodic in each of the ϕ_i with period 2π, and the submanifold $\mathcal{M}_{\vec{f}}$ of phase space which has a given set $\{f_i\}$ of values for the J_i is an n-dimensional torus. As the J_i are conserved, the motion is confined to $\mathcal{M}_{\vec{f}}$, and indeed the equations of motion are very simple, $d\phi_i/dt = \omega_i$ (constant). $\mathcal{M}_{\vec{f}}$ is known as an invariant torus.

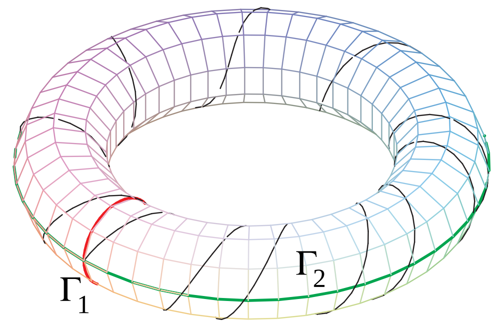


Fig 8. For an integrable system with two degrees of freedom, the motion is confined to a 2-torus, and the trajectories are uniform motion in each of the angles, with independent frequencies. The two actions J_1 and J_2 may be considered as integrals of the single 1-form $\omega_1 = \sum J_i d\phi_i$ over two independant cycles Γ_1 and Γ_2 as shown.

In the one variable case we related the action to the 1-form $p\,dq$. On the invariant torus, the actions are constants and so it is trivially true that $J_i = \oint J_i d\phi_i/2\pi$, where the integral is $\int_0^{2\pi} d\phi_i$ with the other ϕ's held fixed. This might lead one to think about n 1-forms without a sum, but it is more profitable to recognize that the single 1-form $\omega_1 = \sum J_i d\phi_i$ alone contains all of the information we need. First note that, restricted to $\mathcal{M}_{\vec{f}}$, dJ_i vanishes, so ω_1 is closed on $\mathcal{M}_{\vec{f}}$, and its integral is a topological invariant, that is, unchanged under continuous deformations of the path. We can take a set of paths, or cycles, Γ_i, each winding around the torus only in the ϕ_i direction, and we then have $J_i = \frac{1}{2\pi}\int_{\Gamma_i}\omega_1$. The answer is completely independent of where the path Γ_i is drawn on $\mathcal{M}_{\vec{f}}$, as long as its topology is unchanged. Thus the action can be thought of as a function on the simplicial homology H_1 of $\mathcal{M}_{\vec{f}}$. The actions can also be expressed as an integral over a surface Σ_i bounded by the Γ_i, $J_i = \frac{1}{2\pi}\int_{\Sigma_i}\sum dJ_i \wedge d\phi_i$. Notice that this surface Σ_i does not lie on the invariant torus but cuts across it. This formulation has two advantages. First, $\sum dp_i \wedge dq_i$ is invariant under arbitrary canonical transformations, so $\sum dJ_i \wedge d\phi_i$ is just one way to write it. Secondly, on a surface of constant t, such as Σ_i, it is identical to the fundamental form

$$d\omega_3 = \sum_{i=1}^{n} dp_i \wedge dq_i - dH \wedge dt,$$

the generalization to several degrees of freedom of the form we used to show the invariance of the integral under time evolution in the single degree of freedom case.

Now suppose that our system is subject to some time-dependent perturbation, but that at all times its Hamiltonian remains close to an integrable system, though that system might have parameters which vary with time. Let's also assume that after time T the hamiltonian again becomes an autonomous integrable system, though perhaps with parameters different from what it had at $t = 0$.

Consider the evolution in time, under the full hamiltonian, of each system which at $t = 0$ was at some point $\vec{\phi}_0$ on the invariant torus $\mathcal{M}_{\vec{f}}$ of the original unperturbed system. Follow each such system until time T. We assume that none of these systems reaches a critical point during this evolution. The region in phase space thus varies continuously, and at the fixed later time T, it still will be topologically an n-torus, which we will call $\mathcal{B}$. The image of each of the cycles Γ_i will be a cycle $\tilde{\Gamma}_i$ on $\mathcal{B}$, and together these images will be a a basis of the homology H_1 of the $\mathcal{B}$. Let $\tilde{\Sigma}_i$ be surfaces within the $t = T$ hyperplane bounded by $\tilde{\Gamma}_i$. Define $\tilde{J}_i$ to be the integral on $\tilde{\Sigma}_i$ of $d\omega_3$, so $\tilde{J}_i = \frac{1}{2\pi}\int_{\tilde{\Sigma}_i}\sum_j dp_j \wedge dq_j$, where we can drop the $dH \wedge dt$ term on a constant t surface, as $dt = 0$. We can now repeat the argument from the one-degree-of-freedom case to show that the integrals $\tilde{J}_i = J_i$, again because $d\omega_3$ is a closed 2-form which vanishes on the surface of evolution, so that its integrals on the end-caps are the same.

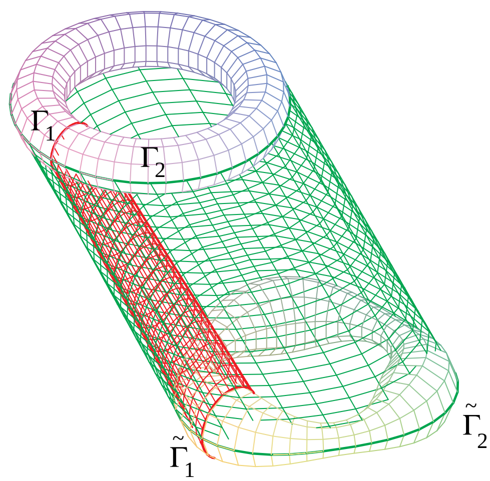


Fig. 9. Time evolution of the invariant torus, and each of two of the cycles on it.

Now we have assumed that the system is again integrable at $t = T$, so there are new actions J'_i, and new invariant tori

$$\mathcal{M}'_{\vec{g}} = \{(\vec{q}, \vec{p}) \ni J'_i(\vec{q}, \vec{p}) = g_i\}.$$

Each initial system which started at $\vec{\phi}_0$ winds up on some new invariant torus with $\vec{g}(\vec{\phi}_0)$.

If the variation of the hamiltonian is sufficiently slow and smoothly varying on phase space, and if the unperturbed motion is sufficiently ergotic that each system samples the full invariant torus on a time scale short compared to the variation time of the hamiltonian, then each initial system $\vec{\phi}_0$ may be expected to wind up with the same values of the perturbed actions, so $\vec{g}$ is independant of $\vec{\phi}_0$. That means that the torus $\mathcal{B}$ is, to some good approximation, one of the invariant tori $\mathcal{M}'_{\vec{g}}$, that the cycles of $\mathcal{B}$ are cycles of $\mathcal{M}'_{\vec{g}}$, and therefore that $J'_i = \tilde{J}_i = J_i$, and each of the actions is an adiabatic invariant.

7.3.5 Formal Perturbative Treatment

Consider a system based on a system $H(\vec{q}, \vec{p}, \vec{\lambda})$, where $\vec{\lambda}$ is a set of parameters, which is integrable for each constant value of $\vec{\lambda}$ within some domain of interest. Now suppose our "real" system is described by the same Hamiltonian, but with $\vec{\lambda}(t)$ a given slowly varying function of time. Although the full Hamiltonian is not invariant, we will show that the action variables are approximately so.

For each fixed value of $\vec{\lambda}$, there is a generating function of type 1 to the corresponding action-angle variables:

$$F_1(\vec{q}, \vec{\phi}, \vec{\lambda}) : (\vec{q}, \vec{p}) \to (\vec{\phi}, \vec{I}).$$

This is a time-independent transformation, so the Hamiltonian may be written as $H(\vec{I}(\vec{q}, \vec{p}), \vec{\lambda})$, independent of the angle variable. This constant $\vec{\lambda}$ Hamiltonian has equations of motion $\dot{\phi}_i = \partial H/\partial I_i = \omega_i(\vec{\lambda}), \dot{I}_i = 0$. But in the case where $\vec{\lambda}$ is a function of time, the transformation F_1 is not a time-independent one, so the correct Hamiltonian is not just the reexpressed Hamiltonian but has an additional term

$$K(\vec{\phi}, \vec{I}, \vec{\lambda}) = H(\vec{I}, \vec{\lambda}) + \sum_n \frac{\partial F_1}{\partial \lambda_n} \frac{d\lambda_n}{dt},$$

where the second term is the expansion of $\partial F_1/\partial t$ by the chain rule. The equations of motion involve differentiating K with respect to one of the variables (ϕ_j, I_j) holding the others, and time, fixed. While these are not the usual variables $(\vec{q}, \vec{\phi})$ for F_1, they are coordinates of phase space, so F_1 can be expressed in terms of (ϕ_j, I_j), and as shown in (7.2), it is periodic in the ϕ_j. The equation of motion for I_j is

$$\begin{aligned} \dot{\phi}_i &= \omega_i(\vec{\lambda}) + \sum_n \frac{\partial^2 F_1}{\partial\lambda_n \partial I_i}\dot{\lambda}_n, \\ \dot{I}_i &= \sum_n \frac{\partial^2 F_1}{\partial\lambda_n \partial\phi_i}\dot{\lambda}_n, \end{aligned}$$

where all the partial derivatives are with respect to the variables $\vec{\phi}, \vec{I}, \vec{\lambda}$. We first note that if the parameters λ are slowly varying, the $\dot{\lambda}_n$'s in the equations of motion make the deviations from the unperturbed system small, of first order in $\epsilon/\tau = \dot{\lambda}/\lambda$, where τ is a typical time for oscillation of the system. But in fact the constancy of the action is better than that, because the expression for $\dot{I}_j$ is predominantly an oscillatory term with zero mean. This is most easily analyzed when the unperturbed system is truly periodic, with period τ. Then during one period $t \in [0, \tau]$, $\dot{\lambda}(t) \approx \dot{\lambda}(0) + t\ddot{\lambda}$. Assuming $\lambda(t)$ varies smoothly on a time scale τ/ϵ, $\ddot{\lambda} \sim \lambda\mathcal{O}(\epsilon^2/\tau^2)$, so if we are willing to drop terms of order ϵ^2, we may treat $\dot{\lambda}$ as a constant. We can then also evaluate F_1 on the orbit of the unperturbed system, as that differs from the true orbit by order ϵ, and the resulting value is multiplied by $\dot{\lambda}$, which is already of order ϵ/τ, and the result is to be integrated over a period τ. Then we may write the change of I_j over one period as

$$\Delta I_j \approx \sum_n \dot{\lambda}_n \int_0^\tau \frac{\partial}{\partial\phi_j}\left(\frac{\partial F_1}{\partial\lambda_n}\right) dt.$$

But F_1 is a well defined single-valued function on the invariant manifold, and so are its derivatives with respect to λ_n, so we may replace the time integral by an integral over the orbit,

$$\Delta I_j \approx \sum_n \dot{\lambda}_n \frac{\tau}{L} \oint \frac{\partial}{\partial\phi_j}\left(\frac{\partial F_1}{\partial\lambda_n}\right) d\phi_j = 0,$$

where L is the length of the orbit, and we have used the fact that for the unperturbed system $d\phi_j/dt$ is constant.

Thus the action variables have oscillations of order ϵ, but these variations do not grow with time. Over a time t, $\Delta\vec{I} = \mathcal{O}(\epsilon)+t\mathcal{O}(\epsilon^2/\tau)$, and is therefore conserved up to order ϵ even for times as large as τ/ϵ, corresponding to many natural periods, and also corresponding to the time scale on which the Hamiltonian is varying significantly.

This form of perturbation, corresponding to variation of constants on a time scale slow compared to the natural frequencies of the unperturbed system, is known as an adiabatic variation, and a quantity conserved to order ϵ over times comparable to the variation itself is called an **adiabatic invariant**. Classic examples include ideal gases in a slowly varying container, a pendulum of slowly varying length, and the motion of a rapidly moving charged particle in a strong but slowly varying magnetic field. It is interesting to note that in Bohr-Sommerfeld quantization in the old quantum mechanics, used before the Schrödinger equation clarified such issues, the quantization of bound states was related to quantization of the action. For example, in Bohr theory the electrons are in states with action nh, with n a positive integer and h Planck's constant. Because these values are preserved under adiabatic perturbation, it is possible that an adiabatic perturbation of a quantum mechanical system maintains the system in the initial quantum mechanical state, and indeed this can be shown, with the full quantum theory, to be the case in general. An important application is cooling by adiabatic demagnetization. Here atoms with a magnetic moment are placed in a strong magnetic field and reach equilibrium according to the Boltzman distribution for their polarizations. If the magnetic field is adiabatically reduced, the separation energies of the various polarization states is reduced proportionally. As the distribution of polarization states remains the same for the adiabatic change, it now fits a Boltzman distribution for a temperature reduced proportionally to the field, so the atoms have been cooled.

7.4 Rapidly Varying Perturbations

At the other extreme from adiabatic perturbations, we may ask what happens to a system if we add a perturbative potential which oscillates rapidly with respect to the natural frequencies of the unperturbed system. If these forces are of the same magnitude as those of the unperturbed system, we would expect that they would cause in the coordinates and momenta a small rapid oscillation, small because a finite acceleration could make only small changes in velocity and position over a small oscillation time. Then we might expect the effects of the force to be little more than adding jitter to the unperturbed motion. Consider the case that the external force is a pure sinusoidal oscillation,

$$H(\vec{q}, \vec{p}) = H_0(\vec{q}, \vec{p}) + U(\vec{q}) \sin \omega t,$$

and let us write the resulting motion as

$$\begin{aligned} q_j(t) &= \bar{q}_j(t) + \xi_j(t), \\ p_j(t) &= \bar{p}_j(t) + \eta_j(t), \end{aligned}$$

where we subtract out the average smoothly varying functions $\bar{q}$ and $\bar{p}$, leaving the rapidly oscillating pieces $\vec{\xi}$ and $\vec{\eta}$, which have natural time scales of $2\pi/\omega$. Thus $\ddot{\xi}, \omega\dot{\xi}, \omega^2\xi, \dot{\eta}$ and $\omega\eta$ should all remain finite as ω gets large with all the parameters of H_0 and $U(q)$ fixed. Our naïve expectation is that the $\bar{q}(t)$ and $\bar{p}(t)$ are what they would have been in the absence of the perturbation, and $\xi(t)$ and $\eta(t)$ are purely due to the oscillating force.

This is not exactly right, however, because the force due to H_0 depends on the q and p at which it is evaluated, and it is being evaluated at the full $q(t)$ and $p(t)$ rather than at $\bar{q}(t)$ and $\bar{p}(t)$. In averaging over an oscillation, the first derivative terms in H_0 will not contribute to a change, but the second derivative terms will cause the average value of the force to differ from its value at $(\bar{q}(t), \bar{p}(t))$. The lowest order effect $(\mathcal{O}(\omega^{-2}))$ is from the oscillation of $p(t)$, with $\eta \propto \omega^{-1}\partial U/\partial q$, changing the average force by an amount proportional to η^2 times $\partial^2 H_0/\partial p_k \partial p_\ell$. We shall see that a good approximation is to take $\bar{q}$ and $\bar{p}$ to evolve with the effective "**mean motion Hamiltonian**"

$$K(\bar{q}, \bar{p}) = H_0(\bar{q}, \bar{p}) + \frac{1}{4\omega^2} \sum_{k\ell} \frac{\partial U}{\partial \bar{q}_k} \frac{\partial U}{\partial \bar{q}_\ell} \frac{\partial^2 H_0}{\partial \bar{p}_k \partial \bar{p}_\ell}. \tag{7.11}$$

Under this hamiltonian, we have

$$\begin{aligned} \dot{\bar{q}}_j &= \frac{\partial K}{\partial p_j} = \left.\frac{\partial H_0}{\partial p_j}\right|_{\bar{q},\bar{p}} + \frac{1}{4\omega^2} \sum_{k\ell} \frac{\partial U}{\partial \bar{q}_k} \frac{\partial U}{\partial \bar{q}_\ell} \frac{\partial^3 H_0}{\partial \bar{p}_k \partial \bar{p}_\ell \partial \bar{p}_j}. \\ \dot{\bar{p}}_j &= -\frac{\partial K}{\partial q_j} \qquad\qquad (7.12) \\ &= -\left.\frac{\partial H_0}{\partial q_j}\right|_{\bar{q},\bar{p}} - \frac{1}{2\omega^2} \sum_{k\ell} \frac{\partial^2 U}{\partial \bar{q}_j \partial \bar{q}_k} \frac{\partial U}{\partial \bar{q}_\ell} \frac{\partial^2 H_0}{\partial \bar{p}_k \partial \bar{p}_\ell} - \frac{1}{4\omega^2} \sum_{k\ell} \frac{\partial U}{\partial \bar{q}_k} \frac{\partial U}{\partial \bar{q}_\ell} \frac{\partial^3 H_0}{\partial \bar{p}_k \partial \bar{p}_\ell \partial \bar{q}_j}. \end{aligned}$$

Of course the full motion for $q(t)$ and $p(t)$ is given by the full Hamiltonian equations:

$$\begin{aligned}
\dot{\bar{q}}_j + \dot{\xi}_j &= \left.\frac{\partial H_0}{\partial p_j}\right|_{q,p} \\
&= \left.\frac{\partial H_0}{\partial p_j}\right|_{\bar{q},\bar{p}} + \sum_k \xi_k \left.\frac{\partial^2 H_0}{\partial p_j \partial q_k}\right|_{\bar{q},\bar{p}} + \sum_k \eta_k \left.\frac{\partial^2 H_0}{\partial p_j \partial p_k}\right|_{\bar{q},\bar{p}} \\
&\quad + \frac{1}{2}\sum_{k\ell} \eta_k \eta_\ell \left.\frac{\partial^3 H_0}{\partial p_j \partial p_k \partial p_\ell}\right|_{\bar{q},\bar{p}} + \mathcal{O}(\omega^{-3}) \\
\dot{\bar{p}}_j + \dot{\eta}_j &= -\left.\frac{\partial H_0}{\partial q_j}\right|_{q,p} - \left.\frac{\partial U}{\partial q_j}\right|_{q,p} \sin\omega t \\
&= -\left.\frac{\partial H_0}{\partial q_j}\right|_{\bar{q},\bar{p}} - \sum_k \xi_k \left.\frac{\partial^2 H_0}{\partial q_j \partial q_k}\right|_{\bar{q},\bar{p}} - \sum_k \eta_k \left.\frac{\partial^2 H_0}{\partial q_j \partial p_k}\right|_{\bar{q},\bar{p}} \\
&\quad - \frac{1}{2}\sum_{k\ell} \eta_k \eta_\ell \left.\frac{\partial^3 H_0}{\partial q_j \partial p_k \partial p_\ell}\right|_{\bar{q},\bar{p}} - \sin\omega t \left.\frac{\partial U}{\partial q_j}\right|_{\bar{q}} \\
&\quad - \sum_k \xi_k \sin\omega t \left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_{\bar{q}} + \mathcal{O}(\omega^{-3}). \qquad (7.13)
\end{aligned}$$

Subtracting (7.12) from (7.13) gives

$$\begin{aligned}
\dot{\xi}_j &= \sum_k \eta_k \left.\frac{\partial^2 H_0}{\partial p_j \partial p_k}\right|_{\bar{q},\bar{p}} + \sum_k \xi_k \left.\frac{\partial^2 H_0}{\partial p_j \partial q_k}\right|_{\bar{q},\bar{p}} + \\
&\quad + \frac{1}{2}\sum_{k\ell}\left(\eta_k\eta_\ell - \frac{1}{2\omega^2}\frac{\partial U}{\partial \bar{q}_k}\frac{\partial U}{\partial \bar{q}_\ell}\right) \left.\frac{\partial^3 H_0}{\partial p_j \partial p_k \partial p_\ell}\right|_{\bar{q},\bar{p}} + \mathcal{O}(\omega^{-3}) \qquad (7.14) \\
\dot{\eta}_j &= -\sin\omega t \left.\frac{\partial U}{\partial q_j}\right|_{\bar{q}} - \sum_k \eta_k \left.\frac{\partial^2 H_0}{\partial q_j \partial p_k}\right|_{\bar{q},\bar{p}} - \sum_k \xi_k \left.\frac{\partial^2 H_0}{\partial q_j \partial q_k}\right|_{\bar{q},\bar{p}} \\
&\quad - \frac{1}{2}\sum_{k\ell}\left(\eta_k\eta_\ell - \frac{1}{2\omega^2}\frac{\partial U}{\partial \bar{q}_k}\frac{\partial U}{\partial \bar{q}_\ell}\right) \left.\frac{\partial^3 H_0}{\partial q_j \partial p_k \partial p_\ell}\right|_{\bar{q},\bar{p}} \qquad (7.15) \\
&\quad - \sum_k \left(\xi_k \sin\omega t - \frac{1}{2\omega^2}\sum_\ell \frac{\partial U}{\partial q_\ell}\frac{\partial^2 H_0}{\partial p_k \partial p_\ell}\right) \left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_{\bar{q}} + \mathcal{O}(\omega^{-3}).
\end{aligned}$$

All variables in expressions (7.14) and (7.15) are evaluated at time t. We wish to show that over a full period $\tau = 2\pi/\omega$, η and ξ grow only negligibly, that is, $\Delta\eta$ and $\Delta\xi$ vanish to $\mathcal{O}(\omega^{-3})$, for which we need the derivatives to order $\mathcal{O}(\omega^{-2})$. During a period, the change in $\bar{q}$ and $\bar{p}$ will be $\mathcal{O}(\omega^{-1})$, so in evaluating the H_0 and U derivative terms in which they are multiplied by things already $\mathcal{O}(\omega^{-2})$, we can treat them as constants.

To lowest order in ω^{-1}, we see that

$$\eta_j(t') = \frac{1}{\omega}\cos\omega t' \left.\frac{\partial U}{\partial q_j}\right|_{\bar{q}} + \text{const} + \mathcal{O}(\omega^{-2}).$$

The ambiguity in the integration constant is an ambiguity in our initial condition for $\bar{p}$, so we can set the constant to zero, or better yet, arranged so that the average value of η_j over one period is zero. So we require $<\eta_k> = 0$. Our expression for $\eta_j(t')$ is good enough to integrate (7.14) for $\xi_j(t')$ to order $\mathcal{O}(\omega^{-3})$,

$$\xi_j(t') = \frac{1}{\omega^2}\sin\omega t' \sum_k \frac{\partial U}{\partial \bar{q}_k}\frac{\partial^2 H_0}{\partial p_j \partial p_k} + \mathcal{O}(\omega^{-3}),$$

where we have again dropped the integration constant as a correction to the initial condition for $\bar{q}$. Notice that the average of ξ_j over one period is zero, to the order required.

Now we are ready to find whether η and ξ change over the course of one period. We will use

$$\begin{aligned}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} \sin\omega t' f(t')\,dt' &= \frac{2\pi}{\omega^2}\frac{df}{dt}\cos\omega t + \mathcal{O}(\omega^{-3})\\ \int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} \cos\omega t f(t)\,dt &= -\frac{2\pi}{\omega^2}\frac{df}{dt}\sin\omega t + \mathcal{O}(\omega^{-3})\end{aligned}$$

In particular,

$$\begin{aligned}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} \sin\omega t' \left.\frac{\partial U}{\partial q_j}\right|_{\bar{q}(t')} dt' &= \frac{2\pi}{\omega^2}\cos\omega t \sum_k \left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_{\bar{q}(t)} \dot{q}_k\\ &= \frac{2\pi}{\omega^2}\cos\omega t \sum_k \left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_{\bar{q}(t)} \left.\frac{\partial H_0}{\partial p_k}\right|_{\bar{q}(t),\bar{p}(t)}.\end{aligned}$$

We also see that

$$\begin{aligned}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} \eta_k(t') f(t')\,dt' &= \int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} \eta_k(t')\left(f(t) + (t'-t)\left.\frac{df}{dt}\right|_t\right) dt' + \mathcal{O}(\omega^{-3})\\ &= \frac{2\pi}{\omega}<\eta_k>f(t) + \left.\frac{df}{dt}\right|_t \int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} (t'-t)\eta_k(t')\,dt'\\ &= \frac{2\pi}{\omega}<\eta_k>f(t) + \mathcal{O}(\omega^{-3})\end{aligned}$$

because $\eta_k(t')$ is already $\mathcal{O}(\omega^{-1})$, is multiplied by something less than τ and integrated over an interval of lenght τ.

So we can write that the changes in η and ξ over one period are

$$\begin{aligned}
\Delta\xi_j &= \int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} \dot{\xi}_j(t')\,dt' \\
&= \frac{2\pi}{\omega}\left[\sum_k <\eta_k> \left.\frac{\partial^2 H_0}{\partial p_j \partial p_k}\right|_{\bar{q},\bar{p}} + \sum_k <\xi_k> \left.\frac{\partial^2 H_0}{\partial p_j \partial q_k}\right|_{\bar{q},\bar{p}} \right. \\
&\qquad \left. + \frac{1}{2}\sum_{k\ell}\left(<\eta_k\eta_\ell> - \frac{1}{2\omega^2}\frac{\partial U}{\partial \bar{q}_k}\frac{\partial U}{\partial \bar{q}_\ell}\right) \left.\frac{\partial^3 H_0}{\partial p_j \partial p_k \partial p_\ell}\right|_{\bar{q},\bar{p}}\right] + \mathcal{O}(\omega^{-4})
\end{aligned}$$

$$\begin{aligned}
\Delta\eta_j &= \int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} \dot{\eta}_j(t')\,dt' \\
&= -\frac{2\pi}{\omega^2}\sum_k \frac{\partial^2 U}{\partial q_j \partial q_k}\frac{\partial H_0}{\partial p_k}\cos\omega t - \frac{2\pi}{\omega}\sum_k <\eta_k> \left.\frac{\partial^2 H_0}{\partial q_j \partial p_k}\right|_{\bar{q},\bar{p}} \\
&\quad -\frac{\pi}{\omega}\sum_{k\ell}\left(<\eta_k\eta_\ell> - \frac{1}{2\omega^2}\frac{\partial U}{\partial \bar{q}_k}\frac{\partial U}{\partial \bar{q}_\ell}\right) \left.\frac{\partial^3 H_0}{\partial q_j \partial p_k \partial p_\ell}\right|_{\bar{q},\bar{p}} \\
&\quad -\frac{2\pi}{\omega}\sum_k\left(<\xi_k \sin\omega t> - \frac{1}{2\omega^2}\sum_\ell \frac{\partial U}{\partial q_\ell}\frac{\partial^2 H_0}{\partial p_k \partial p_\ell}\right) \left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_{\bar{q}} + \mathcal{O}(\omega^{-4}).
\end{aligned}$$

We need

$$\begin{aligned}
<\eta_k\eta_\ell> &= \frac{\omega}{2\pi}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} \frac{1}{\omega^2}\cos^2\omega t' \frac{\partial U}{\partial q_k}\frac{\partial U}{\partial q_\ell}\,dt' = \frac{1}{2\omega^2}\frac{\partial U}{\partial q_k}\frac{\partial U}{\partial q_\ell}, \\
<\xi_k \sin\omega t> &= \frac{\omega}{2\pi}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} \frac{1}{\omega^2}\sin^2\omega t' \sum_k \frac{\partial U}{\partial \bar{q}_k}\frac{\partial^2 H_0}{\partial p_j \partial p_k}\,dt' \\
&= \frac{1}{2\omega^2}\sum_k \frac{\partial U}{\partial \bar{q}_k}\frac{\partial^2 H_0}{\partial p_j \partial p_k}
\end{aligned}$$

These, together with our requirement $<\eta_k> = 0$, show that all the terms vanish except

$$\Delta\eta_j = -\frac{2\pi}{\omega^2}\sum_k \frac{\partial^2 U}{\partial q_j \partial q_k}\frac{\partial H_0}{\partial p_k}\cos\omega t.$$

Thus the system evolves as if with the mean field hamiltonian, with a small added oscillatory motion which does not grow (to order ω^{-2} for $q(t)$) with time.

We have seen that there are excellent techniques for dealing with perturbations which are either very slowly varying modifications of a system which would be integrable were the parameters not varying, or with perturbations which are rapidly varying (with zero mean) compared to the natural motion of the unperturbed system.

Exercises

7.1 Consider the harmonic oscillator $H = p^2/2m + \frac{1}{2}m\omega^2 q^2$ as a perturbation on a free particle $H_0 = p^2/2m$. Find Hamilton's Principle Function $S(q, P)$ which generates the transformation of the unperturbed hamiltonian to Q, P the initial position and momentum. From this, find the Hamiltonian $K(Q, P, t)$ for the full harmonic oscillator, and thus equations of motion for Q and P. Solve these iteratively, assuming $P(0) = 0$, through fourth order in ω. Express q and p to this order, and compare to the exact solution for an harmonic oscillator.

7.2 Consider the Kepler problem in two dimensions. That is, a particle of (reduced) mass μ moves in two dimensions under the influence of a potential

$$U(x, y) = -\frac{K}{\sqrt{x^2 + y^2}}.$$

This is an integrable system, with two integrals of the motion which are in involution. In answering this problem you are expected to make use of the explicit solutions we found for the Kepler problem.

a) What are the two integrals of the motion, F_1 and F_2, in more familiar terms and in terms of explicit functions on phase space.

b) Show that F_1 and F_2 are in involution.

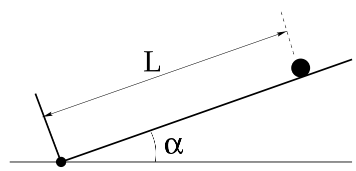
c) Pick an appropriate $\eta_0 \in \mathcal{M}_{\vec{f}}$, and explain how the coordinates $\vec{t}$ are related to the phase space coordinates $\eta = g^{\vec{t}}(\eta_0)$. This discussion may be somewhat qualitative, assuming we both know the explicit solutions of Chapter 3, but it should be clearly stated.

d) Find the vectors $\vec{e}_i$ which describe the unit cell, and give the relation between the angle variables ϕ_i and the usual coordinates η. One of these should be explicit, while the other may be described qualitatively.

e) Comment on whether there are relations among the frequencies and whether this is a degenerate system.

7.3 Consider a mass m hanging at the end of a length of string which passes through a tiny hole, forming a pendulum. The length of string below the hole, $\ell(t)$ is slowly shortened by someone above the hole pulling on the string. How does the amplitude (assumed small) of the oscillation of the pendulum depend on time? (Assume there is no friction).

7.4 A particle of mass m slides without friction on a flat ramp which is hinged at one end, at which there is a fixed wall. When the mass hits the wall it is reflected perfectly elastically. An external agent changes the angle α very slowly compared to the interval between successive times at which the particle reaches a maximum height. If the angle varies from from an initial value of α_I to a final value α_F, and if the maximum excursion is L_I at the beginning, what is the final maximum excursion L_F?



7.5 Consider a particle of mass m and charge q in the field of a fixed electric dipole with moment $\vec{p}$. Using spherical coordinates with the axis in the $\vec{p}$ direction, the potential energy is given by

$$U(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{qp}{r^2}\cos\theta.$$

There is no explicit t or ϕ dependence, so H and $p_\phi = L_z$ are conserved.
a) Show that

$$A = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} + \frac{qpm}{2\pi\epsilon_0}\cos\theta$$

is also conserved.
b) Given these three conserved quantities, what else must you show to find if this is an integrable system? Is it true? What, if any, conditions are there for the motion to be confined to an invariant torus?