Physics 504, Lecture 25
April 28, 2011

1 Bremsstrahlung

We have seen that whenever charged particles are accelerated, radiation will be produced. For elementary particles this effect was first observed when high-energy electrons entered material and quickly lost their energy, which was therefore called braking radiation (in german). For relativistic particles this can be an appreciable deposition of the energy when a particle enters material, in addition to the energy transferred to the particles on which the scattering takes place. Experimentalists use this to measure the energy of electrons, as the radiation produced (as photons) often pair-produce, so the measurement of all the energy of charged particles produced measures that of the original electron.

Another place where understanding bremsstrahlung is important is at the opposite end of the energy spectrum, where we consider low frequency radiation. The process of scattering occurs on an atomic or subatomic scale, and therefore over a very short time interval, so the details of the scattering should be irrelevant when discussing frequencies corresponding to wavelengths larger than an angstrom. Thus in the equation we derived last time,

$$A(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} e^{i\omega R/c} \int_{-\infty}^{\infty} e^{i\omega(t-\hat{n}\cdot\vec{r}(t)/c)} \frac{d}{dt}\left[\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\hat{n}\cdot\vec{\beta}}\right] dt. \tag{1}$$

we can assume that the $d[\,]/dt$ factor contributes over such a small range of t values that the phase $i\omega(t - \hat{n}\cdot\vec{r}(t)/c$ can be taken to be an irrelevant constant phase, and therefore we have an integral of a total derivative, given by the difference of its final and initial values. Projecting the contribution of a single polarization $\vec{\epsilon}$, which is perpendicular to $\hat{n}$, we have

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\vec{\epsilon}^{\,*}\cdot\left(\frac{\vec{\beta}_f}{1-\hat{n}\cdot\vec{\beta}_f} - \frac{\vec{\beta}_i}{1-\hat{n}\cdot\vec{\beta}_i}\right)\right|^2.$$

This formula has a disturbing feature with difficulties at both ends of the frequency spectrum, as it is independent of frequency. This is problematic at the high frequency end because it seems to say that integrating over all frequencies, the total power into any solid angle is infinite. This is not a serious problem, however, because for high frequency our approximation that the acceleration occurs in a time interval small compared to $1/\omega$ is unphysical, and we expect there to be modifications at that end. At the other end, however, our approximation is valid. There is no divergence in the energy radiated, but the quantum-mechanical interpretation is a problem. Quantum mechanically, radiation consists of photons, each of energy $\hbar\omega$, and if we get a classical prediction that we should have an energy deposit of ΔE in a small range of frequencies near ω, we need to interpret that quantum-mechanically that we have an expected number of photons $\Delta E/\hbar\omega$ in that range. But we are predicting energy/unit frequency to be constant and nonzero down to zero frequency, so the expected **number** of photons per energy range diverges as $\omega \to 0$. Usually in quantum physics we ask for the probability that, starting from some initial state, we wind up in a given final state, and here we are finding that the probability that an electron scatters from a nucleus, for example, without any additional particles in the final state, is zero. This is known as infrared divergence, and in quantum field theory we need to take into account that any experimental arrangement has a lowest energy cutoff δE, below which a photon could not have been detected, so when we measure the cross section for elastic scattering $\vec{\beta}_i \to \vec{\beta}_f$, we are actually including processes for which an arbitrary number of very soft photons, with total energy totaling less than δE. It turns out the cross section for this is finite and calculable.

Beyond those difficulties, the most noticeable feature of bremsstrahlung radiation for ultrarelativistic particles is the strong peaking in the directions of the final particle and the initial particle. We can think of these as parts of the Coulomb field that have travelled on without noticing that the particle got bumped. In quantum field theory, what is relevant is that the state of the charged particle, of mass m with momentum p^α, together with the photon of momentum k^α, has a mass

$$\begin{aligned} M^2 &= \frac{1}{c^2}\left(p^\alpha + k^\alpha\right)^2 = m^2 + \frac{2}{c^2} p^\alpha k_\alpha = m^2 + 2\frac{m|k|}{c}\gamma(1-\hat{n}\cdot\vec{\beta}) \\ &\approx m^2 + \frac{m|k|}{\gamma c}(1+\gamma^2\theta^2), \end{aligned}$$

which goes to m^2 (goes on-shell in QFT language) as $\beta \to 1$ and the angle θ between the photon and the charged particle goes to zero. Thus in some ways it acts as if this combination were the single charged particle.

2 Bremsstrahlung in Beta Decay

Beta decay is a weak interaction process in which a neutron or proton in a nucleus emits an electron or positron and an antineutrino or neutrino, turning into a proton or neutron respectively, and thereby changing the charge of the nucleus by one unit

$$Z \to (Z \pm 1) + e^{\mp} + \nu.$$

While the electron or positron was created, and not previously residing in the nucleus, its charge was there, so as far as the electromagnetic field is concerned, we have a charge at rest until the sudden decay, and then a pretty-much instantaneous acceleration to a large final velocity. Assuming the decay takes place at $t = 0$, we have

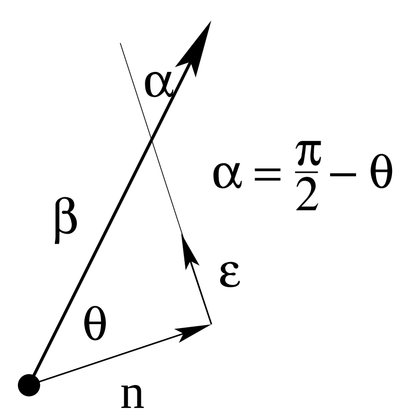
$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c} \left| \frac{\vec{\epsilon}^* \cdot \vec{\beta}}{1 - \hat{n} \cdot \vec{\beta}} \right|^2 ,$$

where $c\vec{\beta}$ is the final velocity of the $e^{\pm}$, and we are assuming ω is small enough. Small enough might mean compared to the scale of the nucleus or nucleon size, as we might imagine the acceleration occurs on this distance scale, but more obviously, if we take into account the fact that any radiation will be quantized in units of $\hbar\omega$, we must have $\hbar\omega < (m_Z - m_{Z\pm 1} - m_e)c^2$.

The radiation in the $\hat{n}$ direction is polarized, as the component perpendicular to the plane containing $\hat{n}$ and $\vec{\beta}$ is not excited (by $\vec{\epsilon}^* \cdot \vec{\beta}$). Then, as we see in the diagram, for the direction that is excited, $\vec{\epsilon}^* \cdot \vec{\beta} = \beta \sin\theta$, and

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c} \beta^2 \frac{\sin^2\theta}{\left(1 - \hat{n} \cdot \vec{\beta}\right)^2}.$$

Integrating over all angles, $\int d\Omega = \int \sin\theta d\theta d\phi = 2\pi \int_{-1}^{1} du$, we have



$$\begin{aligned}\frac{dI}{d\omega} &= \frac{e^2}{4\pi^2 c} 2\pi \int_{-1}^{1} \frac{\beta^2(1-u^2)}{(1-\beta u)^2}\, du = \frac{e^2}{2\pi\beta c}\int_{1-\beta}^{1+\beta} \frac{\beta^2 - 1 + 2x - x^2}{x^2} dx \\ &= \frac{e^2}{\pi c}\left[\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta} - 2\right]\end{aligned}$$

The predicted intensity distribution has no ω dependence, but we have already discussed that there is a cutoff ω_{max}. The energy per frequency value is not large. For small β the term in brackets is $2\beta^2/3$, and we have $dI/d\omega = 2e^2\beta^2/3\pi c$. Let's compare the energy radiated to the lost rest mass energy ΔE which gives the upper limit $\omega_{\text{max}} = \Delta E/\hbar$. Multiplying $dI/d\omega$ by the cutoff frequency we see that the fraction of energy lost going into radiation is less than $2e^2\beta^2/3\pi\hbar c$. As the fine structure constant $\alpha = e^2/\hbar c \approx 1/137$, we see this is well less than 1%. As for bremsstrahlung in scattering, the low frequency limit predicts an infinity at zero frequency in the number spectrum for photons emitted, though the total energy emitted is small.

There is some interest in this lost energy, though it is often ignored, because one way to try to measure neutrino masses is to look for a cutoff in the energy spectrum of the electrons. The energy difference between the parent and daughter nucleus is shared between the electron, the neutrino, and any photons which accompany them. If the neutrino has a mass, there is a lower limit on the energy it can have, and thus an upper limit on the electron energy distribution which is slightly less than $(m_Z - m_{Z\pm 1})c^2$. The expected energy distribution from quantum mechanics, ignoring photon production, might be misleading, however, if inner bremsstrahlung photons are taking off some of the energy.

3 Orbital Electron Capture

From an elementary particle point of view another process which is basically the same, or the inverse of, beta decay, is electron capture. Here a nucleus of charge Ze grabs one of the atomic electrons, converting a proton into a neutron, and emits a neutrino, $Z + e^- \to (Z-1) + \nu$.

Let us first give a classical model for electron capture, and then discuss how to interpret this quantum mechanically. Our classical model has a non-relativistic electron moving in a circle of radius a about the z axis with angular frequency ω_0, up until time $t=0$, at which point it disappears. So for negative t the position and velocity are

$$\begin{aligned}\vec{r}(t) &= a\cos(\omega_0 t+\alpha)\hat{e}_x + a\sin(\omega_0 t+\alpha)\hat{e}_y,\\ \vec{v}(t) &= -\omega_0 a\sin(\omega_0 t+\alpha)\hat{e}_x + \omega_0 a\cos(\omega_0 t+\alpha)\hat{e}_y.\end{aligned}$$

Let's look at this from a point in the xz plane at an angle θ from the z axis, so $\hat{n}=\sin\theta\hat{e}_x+\cos\theta\hat{e}_z$. The frequency spectrum is best evaluated by doing the integration by parts in Eq. 1,

$$\begin{aligned}A(\omega) &= -\sqrt{\frac{q^2}{8\pi^2 c}}e^{i\omega R/c}\int_{-\infty}^{\infty}\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\hat{n}\cdot\vec{\beta}}\frac{d}{dt}\left[e^{i\omega(t-\hat{n}\cdot\vec{r}(t)/c)}\right]dt\\ &= -i\omega\sqrt{\frac{q^2}{8\pi^2 c}}e^{i\omega R/c}\int_{-\infty}^{0}\hat{n}\times(\hat{n}\times\vec{\beta})e^{i\omega t}dt,\end{aligned}$$

where we have assumed the $\omega\hat{n}\cdot\vec{r}(t) < a\omega$ is negligible, and so is the $\vec{\beta}\cdot\hat{n}$. We need $\hat{n}\times(\hat{n}\times\vec{v})$ which is found from $\hat{n}\times\vec{v}=v_x\cos\theta\hat{e}_\perp+v_y\hat{e}_\parallel$, $\hat{n}\times(\hat{n}\times\vec{v}) = -v_x\cos\theta\hat{e}_\parallel+v_y\hat{e}_\perp$, The integrals are given in terms of

$$\begin{aligned}I_1 &= \int_{-\infty}^{0}\cos(\omega_0 t+\alpha)e^{i\omega t} = \frac{-i\omega\cos\alpha-\omega_0\sin\alpha}{\omega^2-\omega_0^2}\\ I_2 &= \int_{-\infty}^{0}\sin(\omega_0 t+\alpha)e^{i\omega t} = \frac{\omega_0\cos\alpha-i\omega\sin\alpha}{\omega^2-\omega_0^2}\end{aligned}$$

Thus

$$\begin{aligned}\frac{d^2I}{d\omega d\Omega} &= \frac{e^2\omega^2\omega_0^2a^2}{4\pi^2c^3}\left|(I_2\cos\theta+I_1)^2+(I_1-I_2\cos\theta)^2\right|\\ &= \frac{e^2\omega^2\omega_0^2a^2}{4\pi^2c^3(\omega^2-\omega_0^2)^2}\left(\omega^2\cos^2\alpha+\omega_0^2\cos^2\alpha+\cos^2\theta(\omega_0^2\cos^2\alpha+\omega^2\cos^2\alpha)\right)\end{aligned}$$

The position in the circle at the moment of absorbtion is random, so we average over the value of α, to get

$$\frac{d^2I}{d\omega d\Omega} = \frac{e^2\omega^2\omega_0^2a^2}{4\pi^2c^3(\omega^2-\omega_0^2)^2}\frac{(\omega^2+\omega_0^2)\left(1+\cos^2\theta\right)}{2}$$

If we don't know in which direction the electron was rotating we should average over angles, or if we want the total power radiated per frequency range, we should integrate:

$$\frac{dI}{d\omega}=\int d\Omega\frac{d^2I}{d\omega d\Omega}=\frac{2e^2}{3\pi c}\left(\frac{\omega_0 a}{c}\right)^2\frac{\omega^2\left(\omega^2+\omega_0^2\right)}{(\omega^2-\omega_0^2)^2}.$$

The quantum mechanical interpretation of this classical model is not completely clear to me. Electron capture requires the electron to be within the nucleus, which is very small compared to atomic orbitals, even in heavy atoms for which the radius is the Bohr radius a_0 divided by Z. Only s wave functions have a nonzero amplitude to be at $r = 0$, so it is an s orbital, usually in the K shell $(n = 1, \ell = 0)$ that is captured, and the circular orbit is not a good description of the spherically symmetric s orbital wave function. On the other hand, very soon after the nucleus gobbles up the innermost electron, an electron in the 2p orbital will make a transition into the newly emptied 1s state, so probably that is what we are describing. The energy emission of a $2p \to 1s$ transition is approximately $3Z^2e^2/8a_0$ which is the best thing to use for $\hbar\omega_0$ in our classical approximation. Here $a_0 = \hbar^2/me^2 = 53$ pm is the Bohr radius. The 2p radius is roughly $a = a_0/Z$. Thus the expected number of photons emitted per energy is

$$N(\hbar\omega) = \frac{dI(\omega)}{\hbar d\omega} \Big/ \hbar\omega \approx \frac{3}{32\pi} Z^2 \left(\frac{e^2}{\hbar c}\right)^3 \frac{1}{\hbar\omega} \left[\frac{\omega^2(\omega^2+\omega_0^2)}{\left(\omega^2-\omega_0^2\right)^2}\right].$$

At high ω the energy distribution goes to a constant, as we have seen for other bremsstrahlung processes with the instantaneous transition approximation, and again we remind ourselves that there is a cutoff in the applicability of our instantaneous change approximation. There is no infrared problem here. The dominant feature is the resonance peak at $\omega = \omega_0$, which is not really infinite as we have it. As we know, the classical picture of a periodic orbit lasting forever would radiate at a constant rate, or infinitely much in infinite time, but quantum mechanics prevents such decays as long as there is no lower-energy state for the electron to fall into. The spread of this peak is due to the finite time it takes, on average, for the transition to take place.

There is also a contribution from the magnetic dipole moment of the electron, but I leave that to Jackson.