Physics 504, Lecture 24
April 25, 2011

1 Frequency and Angular Distribution

We have found the expression for the power radiated in a given solid angle, as a function of time, to be

$$\frac{dP(t)}{d\Omega} = |\vec{A}(t)|^2 \quad \text{where } \vec{A}(t) := \sqrt{\frac{c}{4\pi}}\left[R\,\vec{E}\right]_{\text{ret}}.$$

[Note $\vec{A}$ is **not** the vector potential here!] The energy into a solid angle, over all times, is

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} |A(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{A}(\omega)|^2 d\omega,$$

where $\tilde{A}(\omega)$ is the Fourier transform of $A(t)$,

$$\tilde{A}(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{A}(t)\, e^{i\omega t} dt.$$

As $\vec{A}(t)$ is real, $\tilde{A}(-\omega) = (\tilde{A}(\omega))^*$, so

$$\frac{dW}{d\Omega} = 2\int_0^{\infty} |\tilde{A}(\omega)|^2 d\omega,$$

and we can define the energy per unit solid angle per unit frequency,

$$\frac{d^2 I}{d\omega d\Omega} = 2|\vec{A}(\omega)|^2.$$

Our expression for the radiative part of the electric field,

$$R\vec{E}(t) = \frac{q}{c}\left.\frac{\hat{n}\times\left((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{(1-\hat{n}\cdot\vec{\beta})^3}\right|_{t_e}.$$

$$A(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}}\int_{-\infty}^{\infty} e^{i\omega t}\left[\frac{\hat{n}\times\left((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{(1-\hat{n}\cdot\vec{\beta})^3}\right]_{t_e} dt$$

where $t = t_e + R(t_e)/c$, $dt/dt_e = 1 + (dR/cdt_e) = 1 - \hat{n}\cdot\vec{\beta}(t_e)$, So expressing the integral over t_e, we have

$$A(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} e^{i\omega(t_e+R(t_e)/c)} \left[\frac{\hat{n}\times\left((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{(1-\hat{n}\cdot\vec{\beta})^2}\right] dt_e,$$

and now that there are not references to t left we can drop the subscript e. Assuming the region in which $\dot{\vec{\beta}}$ is nonzero is small compared to R, we can write $R(t) = R - \hat{n}\cdot\vec{r}(t)$, where the observer is a distance R from the origin, which is near the region where the scattering occurs, and $\vec{r}(t)$ is the position of the particle relative to that origin. Then

$$A(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} e^{i\omega R/c} \int_{-\infty}^{\infty} e^{i\omega(t-\hat{n}\cdot\vec{r}(t)/c)} \left[\frac{\hat{n}\times\left((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{(1-\hat{n}\cdot\vec{\beta})^2}\right] dt.$$

In calculating $d^2I/d\omega d\Omega$ the phase factor $e^{i\omega R/c}$ will be irrelevant. We note that the piece in the integrand multiplying the exponential can be written as a total time derivative:

$$\begin{aligned}
\frac{d}{dt}\left[\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\hat{n}\cdot\vec{\beta}}\right] &= \frac{\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})}{1-\hat{n}\cdot\beta} + \frac{\hat{n}\times(\hat{n}\times\vec{\beta})(\hat{n}\cdot\dot{\vec{\beta}})}{(1-\hat{n}\cdot\beta)^2} \\
&= \frac{[(\hat{n}\cdot\dot{\vec{\beta}})\hat{n}-\dot{\vec{\beta}}](1-\hat{n}\cdot\beta) + [(\hat{n}\cdot\beta)\hat{n}-\vec{\beta}](\hat{n}\cdot\dot{\vec{\beta}})}{(1-\hat{n}\cdot\beta)^2} \\
&= \frac{(\hat{n}\cdot\dot{\vec{\beta}})(\hat{n}-\vec{\beta}) - \dot{\vec{\beta}}(1-\hat{n}\cdot\beta)}{(1-\hat{n}\cdot\beta)^2} \\
&= \frac{\hat{n}\times\left((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{(1-\hat{n}\cdot\vec{\beta})^2}.
\end{aligned}$$

Thus we have

$$A(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} e^{i\omega R/c} \int_{-\infty}^{\infty} e^{i\omega(t-\hat{n}\cdot\vec{r}(t)/c)} \frac{d}{dt}\left[\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\hat{n}\cdot\vec{\beta}}\right] dt. \tag{1}$$

It may be useful to integrate by parts, but we will also see, when we discuss the low frequency limit of bremsstrahlung, that this is useful as is.

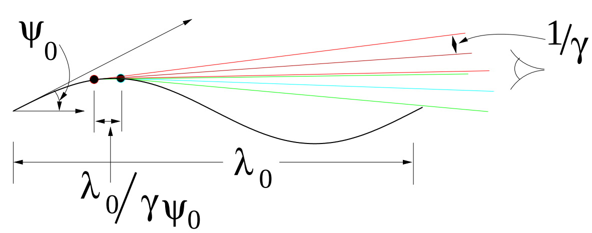
Integrating by parts, assuming that boundary terms at $t = \pm\infty$ can be discarded, and inserting in the intensity, we have

$$\frac{d^2 I}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} e^{i\omega(t-\hat{n}\cdot\vec{r}(t)/c)}\,\hat{n}\times\left(\hat{n}\times\vec{\beta}\right)(t)\,dt\right|^2.$$

2 Wigglers and Undulators

We saw that the pulse of radiation received by an observer from an ultrarelativistic charged particle undergoing transverse acceleration consists of many frequencies up to an X-ray cutoff. This unintentional effect of early high energy accelerators was tapped into by condensed matter experimentalists and biologists who could make use of very intense short pulses of X-rays. But for many purposes a monochromatic rather than broad-spectrum source would be useful.

Enhanced radiation is also possible if you want it. To achieve this, one can produce periodic motion of the particles with a sequence of magnets, called either wigglers or undulators, depending on how significant the oscillations are. A sequence of alternately directed magnets can produce a charged particle path with transverse sinusoidal oscillations, $x = a\sin 2\pi z/\lambda_0$. The angle of the beam will vary by $\psi_0 = \Delta\theta = dx/dz = 2\pi a/\lambda_0$. The spread in angle of the forward radiation is $\theta_r \approx 1/\gamma$, centered on the momentary direction of the beam. Thus if $\psi_0 \gg \theta_r$, an observer will be within the field only part of each oscillation, and will see the source turning on and off. In this case we have a **wiggler**. At the source, that frequency is $\beta c/\lambda_0$. Each wiggle sends a pulse of time duration a fraction roughly (θ_r/ψ_0) of one period, so

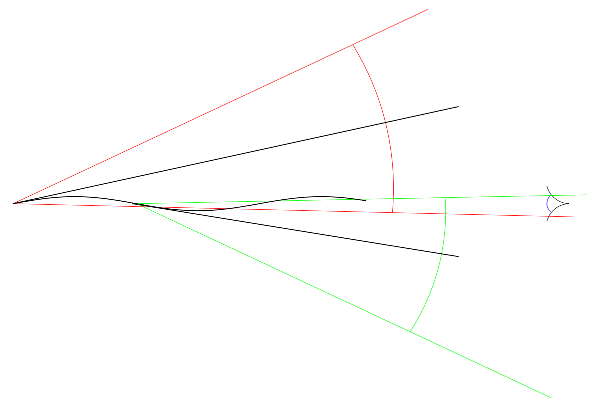


$$\Delta t_e \approx (\lambda_0/\beta c)(\theta_r/\psi_0),$$

but this is compressed for the observer by a factor of $1 - \hat{n}\cdot\vec{\beta} \approx 1/2\gamma^2$, so the received pulse has $\Delta t = \lambda_0/2c\beta\gamma^3\psi_0$ and has frequencies up to $f \approx \frac{1}{\Delta t} \approx 2\gamma^3\psi_0 c/\lambda_0$. Each pulse is incoherent, so the intensity is N times that of a single wiggle.

In the other limit, $\psi_0 \ll \theta_r$, the observer is always in the intense region of the beam, but the beam is radiating coherently. In the particle's rest frame the disturbing fields have a Fitzgerald-contracted wavelength λ_0/γ, going by at βc, so the particle sees itself oscillating at

$$\omega' = 2\pi c\gamma\beta/\lambda_0 \approx 2\pi c\gamma/\lambda_0.$$



But the observer in the lab would say the particle's clock is running slow and therefore the source frequency is ω'/γ, but the Doppler contraction of the pulse increases the frequency by $1/(1-\hat{n}\cdot\vec{\beta}) \approx 2\gamma^2/(1+\gamma^2\theta^2)$. So all together the frequency observed is

$$\omega = \frac{2\omega'}{\gamma(1-\hat{n}\cdot\vec{\beta})} = \frac{4\pi c\gamma^2}{\lambda_0(1+\gamma^2\theta^2)}.$$

Note this is coherent radiation, so the intensity is proportional to N^2 and the frequency has a spread proportional to $1/N$

We will be content with this rather qualitative discussion and skip the fine details of pp 686-694.

3 Thomson Scattering

We saw (14.18) that in the particle's rest frame the electric field is given by

$$\vec{E} = \frac{q}{c^2 R}\,\hat{n}\times(\hat{n}\times\dot{\vec{v}}),$$

so the amplitude corresponding to a particular polarization vector $\vec{\epsilon}$ is

$$\vec{\epsilon}^{\,*}\cdot\vec{E} = \frac{q}{c^2 R}\,\vec{\epsilon}^{\,*}\cdot\left(\hat{n}\times(\hat{n}\times\dot{\vec{v}})\right) = \frac{q}{c^2 R}\,\vec{\epsilon}^{\,*}\cdot\dot{\vec{v}},$$

as $\vec{\epsilon}^{\,*}\cdot\hat{n} = 0$. The power radiated with this polarization per sterradian is

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\left|\vec{\epsilon}^{\,*}\cdot\dot{\vec{v}}\right|^2.$$

If a free electron has an electric field

$$\vec{E}(\vec{x},t) = \vec{\epsilon}_0 E_0 e^{i\vec{k}\cdot\vec{x}-i\omega t}$$

incident on it, it will have an acceleration

$$\dot{\vec{v}}(t) = \vec{\epsilon}_0 \frac{e}{m} E_0 e^{i\vec{k}\cdot\vec{x}-i\omega t}$$

If the motion is sufficiently limited to ignore the change in position and keep the particle non-relativistic, $(x \approx eE_0/m\omega^2 \ll \lambda = 2\pi c/\omega)$, the time average of $\left|\vec{\epsilon}^* \cdot \dot{\vec{v}}\right|^2 = (\vec{\epsilon}^* \cdot \dot{\vec{v}})(\dot{\vec{v}}^* \cdot \vec{\epsilon})$ is

$$\frac{1}{2}\frac{e^2|E_0|^2}{m^2}\left|\vec{\epsilon}^* \cdot \epsilon_0\right|^2$$

and

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi}|E_0|^2 \left(\frac{e^2}{mc^2}\right)^2 \left|\vec{\epsilon}^* \cdot \vec{\epsilon}_0\right|^2 .$$

Dividing this by the incident energy flux $c|E_0|^2/8\pi$ we get the cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \left|\vec{\epsilon}^* \cdot \vec{\epsilon}_0\right|^2 .$$

If the scattering angle is θ and the incident beam is unpolarized and the cross section summed over final polarizations, the factor of

$$\begin{aligned}
\frac{1}{2}\sum_i \sum_f \left|\vec{\epsilon}_f^* \cdot \vec{\epsilon}_0\right|^2 &= \frac{1}{2\pi^2}\int_0^{2\pi} d\phi_i \int_0^{2\pi} d\phi_f [(\cos\theta\cos\phi_f, \sin\phi_f, -\sin\theta\cos\phi_f) \\
&\quad \cdot(\cos\phi_i, \sin\phi_i, 0)]^2 \\
&= \frac{1}{2\pi^2}\int_0^{2\pi} d\phi_i \int_0^{2\pi} d\phi_f \left[(\cos\theta\cos\phi_f\cos\phi_i + \sin\phi_f\sin\phi_i\right]^2 \\
&= \frac{1}{2}\left[\cos^2\theta + 1\right]
\end{aligned}$$

where I have taken the incident direction to be z and the final $(\sin\theta, 0, \cos\theta)$. Thus the unpolarized cross section is

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \frac{1+\cos^2\theta}{2}.$$

This is called the **Thomson formula**. The corresponding total cross section is

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 .$$

The quantity in parentheses is called the **classical electron radius**, roughly the radius at which a conducting sphere of charge e would have electrostatic energy $e^2/2r = mc^2$. (The factor of 1/2, or of 3/5 for a uniformly charged sphere, is discarded.)

This formula disregarded recoil of the electron when hit by the electromagnetic wave. Of course classically the cross section could have been measured with an arbitrarily weak field, so recoil could be neglected, but quantum-mechanically the minimum energy hitting the electron is $\hbar\omega$, which gives a significant recoil if $\hbar\omega \approx mc^2$. In fact, if we take quantum mechanics into account we are considering Compton scattering, for which, we learned as freshman, energy and momentum conservation insure that the outgoing photon has a increased wavelength,

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta), \quad \text{or} \quad \frac{k'}{k} = \frac{1}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos^2\theta)}.$$

It turns out that the quantum mechanical calculation (for a scalar particle) is the classical result times $(k'/k)^2$:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{QM, scalar}} = \left(\frac{e^2}{mc^2}\right)^2 \left(\frac{k'}{k}\right)^2 \left|\vec{\epsilon}^{\,*} \cdot \vec{\epsilon}_0\right|^2 .$$