Physics 504, Lecture 23
April 21, 2011

1 Energy Loss in Accelerators

We found that the power radiated by a relativistic particle is given by Liénard,

$$P = \frac{2}{3}\frac{q^2}{c}\gamma^6\left[(\dot{\vec\beta})^2 - (\vec\beta\times\dot{\vec\beta})^2\right].$$

This is an issue for high-energy accelerators. There are two main types, linear and circular.

In a linear accelerator the direction of $\vec\beta$ is constant so $\dot{\vec\beta} \parallel \vec\beta$ and

$$\begin{aligned} P &= \frac{2e^2}{3c}\gamma^6(\dot\beta)^2 = \frac{2e^2}{3c}\frac{(\dot\gamma)^2}{\beta^2} = \frac{2e^2}{3m^2c^5}\left(\frac{dE}{dt}\right)^2 \Big/ \left(\frac{dx}{cdt}\right)^2 \\ &= \frac{2e^2}{3m^2c^3}\left(\frac{dE}{dx}\right)^2. \end{aligned}$$

So the power radiated is independent of the energy and depends only on the rate of energy change.

The ratio of power lost to power input, dE/dt, for an electron, is

$$\frac{P}{dE/dt} = \frac{2e^2}{3m^2c^3}\frac{1}{v}\frac{dE}{dx} \to \frac{2}{3}\frac{r_e}{mc^2}\frac{dE}{dx},$$

where $r_e = e^2/mc^2$ is the *classical radius of the electron*, 2.82 fm. As we are unlikely to increase the energy of an electron by its rest energy in a distance of its tiny classical radius, energy loss in a linac is negligible.

1.1 Circular Acclerators

In a synchrotron, particles are in circular orbit with the energy changing slowly but the direction of the momentum changing rapidly, $\dot{\vec\beta} = \vec\omega\times\vec\beta \perp \vec\beta$, so

$$P = \frac{2}{3}\frac{q^2}{c}\gamma^6\left[(\dot{\vec\beta})^2 - (\vec\beta\times\dot{\vec\beta})^2\right] = \frac{2}{3}\frac{q^2}{c}\gamma^6\omega^2\beta^2\left[1-\beta^2\right] = \frac{2}{3}\frac{q^2c}{\rho^2}\gamma^4\beta^4,$$

where ρ is the orbit radius and we used $c\beta = \omega\rho$.

The energy loss per revolution δE is the integral of this over an orbit, $\delta t = 2\pi/\omega = 2\pi\rho/c\beta$, or

$$\delta E = \frac{4\pi}{3}q^2\beta^3\gamma^4\left[\rho\left\langle\frac{1}{\rho^2}\right\rangle\right]$$

where the expression in braces is given rather than a simple $1/\rho$ in case you design your accelerator to have magnets not completely covering the circumference[1]

For ultrarelativistic particles, $\beta \to 1$,

$$\delta E \propto E^4/\rho,$$

with the proportionality constant 8.85×10^{-5}m/GeV^4 for electrons and 7.80×10^{-18}m/GeV^4 for protons.

For Lep, an electron beam of roughly 80 GeV and a radius of about 4 km, the electrons lose nearly a GeV per turn! This is why the ring is as big as it is.

For the LHC, which will have 7 TeV protons at the same radius, I get a loss of only 4 KeV per turn, so energy loss is not the crucial issue for proton synchrotrons, but bending radius is.

2 Angular Distribution of Power

We derived he complete expression for $F^{\alpha\beta}$ in covariant form

$$F^{\alpha\beta} = \frac{q}{U_\rho(x^\rho - r^\rho(\tau))}\,\frac{d}{d\tau}\left[\frac{(x-r(\tau))^\alpha U^\beta(\tau) - (x-r(\tau))^\beta U^\alpha(\tau)}{U_\mu(x^\mu - r^\mu(\tau))}\right]\Bigg|_{\tau_0}. \quad (1)$$

but it is often useful to have the expression more explicitly and in three dimensional language. Using $\vec R$ as the 3-vector from $r^\alpha(\tau_0)$ to x^α, with magnitude R and direction $\hat n$, we have $R^\alpha := x^\alpha - r^\alpha(\tau_0) = (R, R\hat n)$, $U^\alpha(\tau_0) = (\gamma c, \gamma c\vec\beta)$

$$\frac{dU^\alpha}{d\tau} = \gamma\frac{dU^\alpha}{dt} = \left(\gamma^4 c\beta\dot\beta, c\gamma^2\dot{\vec\beta} + c\gamma^4\beta\dot\beta\vec\beta\right),$$

[1] See footnote lecture 18, page 2.

where we used $\dot\gamma = \gamma^3\beta\dot\beta$, and we understand that $\vec\beta$, $\dot{\vec\beta}$ and γ are to be evaluated at the retarded time τ_0. In (1) we then have $d(x^\alpha - r^\alpha(\tau)/d\tau = -U^\alpha$ and $U_\rho(x^\rho - r^\rho(\tau_0)) = R\gamma c(1-\hat n\cdot\vec\beta)$, but

$$\frac{d}{d\tau}U\cdot(x-r) = -U^2 + (x-r)_\alpha\frac{dU^\alpha}{d\tau} = -c^2 + R\left(c\gamma^4\beta\dot\beta - \hat n\cdot\dot{\vec\beta}c\gamma^2 - \hat n\cdot\vec\beta c\gamma^4\beta\dot\beta\right).$$

Thus

$$\begin{aligned} F^{\alpha\beta} &= \frac{q}{R^3\gamma^3c^3(1-\hat n\cdot\vec\beta)^3}\Bigg[\left(R^\alpha\frac{dU^\beta}{d\tau} - R^\beta\frac{dU^\alpha}{d\tau}\right)Rc\gamma(1-\hat n\cdot\vec\beta) \\ &\qquad -(R^\alpha U^\beta - R^\beta U^\alpha)\left(-c^2 + R\left\{c\gamma^4\beta\dot\beta - \hat n\cdot\dot{\vec\beta}c\gamma^2 - \hat n\cdot\vec\beta c\gamma^4\beta\dot\beta\right\}\right)\Bigg].\end{aligned}$$

For the electric field,

$$\begin{aligned} \vec E &= -F^{0i}\hat e_i = \frac{q}{R^3\gamma^3c^3(1-\hat n\cdot\vec\beta)^3}\Bigg[\left(R\hat n\gamma^4c\beta\dot\beta - R(c\gamma^2\dot{\vec\beta} + c\gamma^4\beta\dot\beta\vec\beta)\right) \\ &\qquad \cdot Rc\gamma(1-\hat n\cdot\vec\beta) \\ &\qquad -(\gamma cR\hat n - R\gamma c\vec\beta)\left(-c^2 + R\left\{c\gamma^4\beta\dot\beta - \hat n\cdot\dot{\vec\beta}c\gamma^2 - \hat n\cdot\vec\beta c\gamma^4\beta\dot\beta\right\}\right)\Bigg] \\ &= \frac{q}{R^2\gamma^2c^2(1-\hat n\cdot\vec\beta)^3}\Bigg[\left(\gamma^3\beta\dot\beta(\hat n - \vec\beta) - \gamma\dot{\vec\beta}\right)Rc\gamma(1-\hat n\cdot\vec\beta) \\ &\qquad -(\hat n - \vec\beta)\left(-c^2 + R\left\{c\gamma^4\beta\dot\beta(1-\hat n\cdot\vec\beta) - \hat n\cdot\dot{\vec\beta}c\gamma^2\right\}\right)\Bigg] \\ &= \frac{q(\hat n - \vec\beta)}{R^2\gamma^2(1-\hat n\cdot\vec\beta)^3} + \frac{q}{Rc(1-\hat n\cdot\vec\beta)^3}\left(\hat n\cdot\dot{\vec\beta}(\hat n-\vec\beta) - (1-\hat n\cdot\vec\beta)\dot{\vec\beta}\right) \\ &= \frac{q(\hat n - \vec\beta)}{R^2\gamma^2(1-\hat n\cdot\vec\beta)^3} + \frac{q}{Rc}\frac{\hat n\times\left((\hat n-\vec\beta)\times\dot{\vec\beta}\right)}{(1-\hat n\cdot\vec\beta)^3}.\end{aligned}$$

For the magnetic field,

$$\begin{aligned} B_i &= -\frac12\epsilon_{ijk}F^{jk} = -\frac{q\epsilon_{ijk}}{R^3\gamma^3c^3(1-\hat n\cdot\vec\beta)^3} \\ &\qquad \Big[(Rn_jc\gamma^2\dot\beta_k + Rn_jc\gamma^4\beta\dot\beta\beta_k)Rc\gamma(1-\hat n\cdot\vec\beta) \\ &\qquad -Rn_j\gamma c\beta_k\left(-c^2 + R(c\gamma^4\beta\dot\beta(1-\hat n\cdot\vec\beta) - \hat n\cdot\dot{\vec\beta}c\gamma^2\right)\end{aligned}$$

$$\begin{aligned} &= -\frac{q(\hat n\times\vec\beta)_i}{R^2\gamma^2(1-\hat n\cdot\vec\beta)^3} \\ &\qquad -\frac{q}{Rc(1-\hat n\cdot\vec\beta)^3}\left[\hat n\times\dot{\vec\beta}(1-\hat n\cdot\vec\beta) + \hat n\times\vec\beta\hat n\cdot\dot{\vec\beta}\right]_i \\ \text{so}\quad \vec B &= \hat n\times\vec E.\end{aligned}$$

as it should be for a radiation-zone field. Thus we can derive the expression for the power radiated towards the observer, the flux being given by the Poynting vector

$$\hat n\cdot\vec S = \frac{c}{4\pi}E^2.$$

At large distances this is

$$\hat n\cdot\vec S\Big|_{\rm ret} = \frac{q^2}{4\pi cR^2}\left\{\frac{\hat n\times\left[\left(\hat n-\vec\beta\right)\times\dot{\vec\beta}\right]}{\left(1-\hat n\cdot\vec\beta\right)^3}\Bigg|_{\rm ret}\right\}^2.$$

For ultrarelativistic $\beta\approx 1$ particle, radiation very strongly peaks in the near forward direction, as $\hat n\cdot\vec\beta$ can get very close to 1, and the flux received $\propto\left(1-\hat n\cdot\vec\beta\right)^{-6}$. But the power radiated is only $\propto\left(1-\hat n\cdot\vec\beta\right)^{-5}$. Why? $\hat n\cdot\vec S\Big|_{\rm ret}$ is evaluated at our time t, or over some time interval $[t, t+\Delta t]$, but the right hand side is evaluated at our time t_e corresponding to τ_0, the time the light was emitted that is being recieved now. As the light received at $t+\Delta t$ was emitted Δt_e later than the light received at t, it had to travel a distance $\hat n\cdot\vec v\Delta t_e$ less than the light received at t, so $\Delta t = (1-\hat n\cdot\vec\beta)\Delta t_e$. As the total energy emitted is the energy received (we are not changing reference frames here), the power emitted is $(1-\hat n\cdot\vec\beta)$ times the power received.

This is relevant because things are changing with time. If we are dealing with circular motion, the direction of motion is changing, so we are only in the forward direction briefly. It is better to express things in terms of the emission time, t_e, with $t = t_e + R(t_e)/c$ Thus the energy per unit area we receive is

$$E = \int dt\,\hat n\cdot\vec S\Big|_{\rm ret} = \int dt_e\,\hat n\cdot\vec S\Big|_{t_e}\frac{d}{dt_e}\left(t_e + \frac{R(t_e)}{c}\right) = \int dt_e\,\hat n\cdot\vec S\Big|_{t_e}\left(1-\hat n\cdot\vec\beta\right).$$

So the expression which determines the energy distribution is

$$\frac{dP}{dA} = \frac{q^2}{4\pi c R^2} \frac{\left(\hat n \times \left[\left(\hat n - \vec\beta\right) \times \dot{\vec\beta}\right]\right)^2}{\left(1 - \hat n \cdot \vec\beta\right)^5}. \tag{2}$$

Let us consider two important special cases. The first has the acceleration in the same direction as the motion, $\dot{\vec\beta} \parallel \vec\beta$. Then the numerator is $(\hat n \times (\hat n \times \dot{\vec\beta})^2 = \sin^2\theta \dot v^2/c^2$, and

$$\frac{dP}{dA} = \frac{q^2 \dot v^2}{4\pi c^3 R^2} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}.$$

For β close to 1 this is very strongly peaked in the forward direction. The maximum intensity is when

$$\frac{d}{dx}\left(\frac{1-x^2}{(1-\beta x)^5}\right)\Bigg|_{x=\cos\theta} = 0 = \frac{-2x}{(1-\beta x)^5} + \frac{5\beta(1-x^2)}{(1-\beta x)^6}$$

so

$$x = \cos\theta_{\rm max} = \frac{\sqrt{1+15\beta^2}-1}{3\beta}.$$

With $\beta = \sqrt{1-\gamma^{-2}} \to 1 - 1/(2\gamma^2)$, $x \to 1 - \frac{1}{8\gamma^2}$ and $\theta_{\rm max} \to 1/2\gamma$. For such small angles, with $\theta << 1$ but without taking $\gamma\theta$ small, the intensity is

$$\frac{dP}{dA} = \frac{8q^2\dot v^2}{\pi c^3 R^2}\gamma^8 \frac{(\gamma\theta)^2}{(1+\gamma^2\theta^2)^5}.$$

As an example, consider the linear accelerator at SLAC, which accelerates electrons to 50 GeV over a distance of 3 km. At the end, $\gamma_f = 50\text{GeV}/0.511\text{MeV} \approx 10^5$, and, as the travel has been vitually at the speed of light, $\Delta t = 10^{-5}$ s. Assuming the energy gain per meter is constant, this is

$$m_e c^2 \frac{d\gamma}{dt} = m_e c^2 \gamma^3 \beta\dot\beta = m_e c^2 \gamma_f/\Delta t,$$

so the final value of $\dot\beta$ is $1/\gamma_f^2 \Delta t = 10^{-5}/s$. The angle of maximum intensity is $\theta_{\rm max} = 1/200,000$ rad $= 4.1$ seconds of arc, and the power per sterradian from this single electron at that angle is

$$\frac{2^{11}}{5^5\pi}\frac{e^2\dot\beta^2}{c}\gamma^8 = 1.6\times 10^{12}\ \text{W},$$

just from one electron. How can it be so large?

Note this is not the power, it is the power/sterradian. The total power is

$$\begin{aligned} P &\approx 2\pi R^2 \int_0 \theta\, d\theta \frac{dP}{dA} = \frac{16 q^2 \dot v^2}{c^3}\gamma^8 \int_0 \theta\, d\theta \frac{(\gamma\theta)^2}{(1+\gamma^2\theta^2)^5} \\ &= \frac{8q^2\dot v^2}{c^3}\gamma^6 \int_0 du \frac{u}{(1+u)^5} = \frac{2q^2}{3c^3}\dot v^2\gamma^6. \end{aligned}$$

Another important special case is a circular storage ring, where the acceleration is perpendicular to the velocity. Taking $\vec\beta$ in the z direction and $\dot{\vec\beta}$ in the x, and using the usual spherical angles for $\hat n = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, we may evaluate the numerator of (2) as

$$\begin{aligned} \left(\hat n \times [(\hat n - \vec\beta)\times\dot{\vec\beta}]\right)^2 &= \left(\hat n\cdot\dot{\vec\beta}(\hat n - \vec\beta) - [\hat n\cdot(\hat n - \vec\beta)]\dot{\vec\beta}]\right)^2 \\ &= (\hat n\cdot\dot{\vec\beta})^2(\hat n - \vec\beta)^2 - 2(1-\beta\cos\theta)(\hat n - \vec\beta)\cdot\dot{\vec\beta}(\hat n\cdot\dot{\vec\beta}) \\ &\qquad +(1-\beta\cos\theta)^2(\dot{\vec\beta})^2 \\ &= (\hat n\cdot\dot{\vec\beta})^2(1 - 2\beta\cos\theta + \beta^2 - 2(1-\beta\cos\theta)) \\ &\qquad +(1-\beta\cos\theta)^2(\dot{\vec\beta})^2 \\ &= \left[(\sin\theta\cos\phi)^2(-\gamma^{-2}) + (1-\beta\cos\theta)^2\right](\dot{\vec\beta})^2 \end{aligned}$$

so

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\frac{(\dot{\vec v})^2}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right].$$

Again this is strongly peaked in the forward direction. If we take $\theta << 1$, but keeping $\gamma\theta$ to all orders, so $1 - \beta\cos\theta \approx (1+\gamma^2\theta^2)/2\gamma^2$, we have

$$\frac{dP}{d\Omega} \approx \frac{2e^2}{\pi c^3}\frac{\gamma^6(\dot{\vec v})^2}{(1+\gamma^2\theta^2)^3}\left[1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta)^2}\right].$$

The total power radiated in all directions is, from Liénard,

$$P = \frac{2}{3}\frac{e^2}{c^3}(\dot{\vec v})^2\gamma^4,$$

as $(\dot{\vec\beta})^2 - (\vec\beta\times\dot{\vec\beta})^2 = (\dot{\vec\beta})^2(1-\vec\beta^2) = \gamma^{-2}(\dot{\vec\beta})^2$. But do not be mislead into thinking this is weaker than in the case with $\dot{\vec\beta}\parallel\vec\beta$, where we had γ^6 instead of γ^4, because it is very hard to accelerate in the direction of β. A 4-force F in the direction of $\vec{\beta}$ produces $dmc\beta\gamma/d\tau = F = mc\dot{\beta}(\gamma + \beta^2\gamma^3) = mc\dot{\beta}\gamma^3$, so $\dot{\beta} = F/mc\gamma^3$, while a force in the transverse direction has $mc\gamma\dot{\beta} = F$, or $\dot{\beta} = F/mc\gamma$. So the $(\dot{\vec{\beta}})^2$ is likely to be γ^4 bigger in the transverse case. In particular, at the LHC, with 7 TeV protons travelling at roughly c around a 4.3 km radius circle have $\dot{\beta} = \omega \times \beta = 1.1 \times 10^4/$ s, 10^9 times bigger than the electrons at SLAC, even though their γ is a factor of 13 smaller than the γ of the electrons.

Note that for a given size ring, with ultrarelativistic particles travelling at essentially c, the angular velocity and therefore $\dot{\vec{v}}$ is fixed, so the power radiated is proportional to γ^4 or, for a fixed kind of particle, to E^4. This becomes a serious problem at large energies, especially for electrons (as the power radiated is independent of mass for fixed γ).

2.1 Frequency of radiation

Consider a particle in ultrarelativistic circular motion, with radius ρ. As its radiation is essentially confined to a direction $\delta\theta = 1/\gamma$, the arc of the circle during which it irradiates a given distant observer is of length $d = \rho/\gamma$, which it does in a time $\delta t = \rho/\gamma v$. This pulse of light has its leading edge travelling towards the observer a distance $D = c\rho/\gamma v$ during this time, while the trailing edge of the pulse is emitted at d, so the pulse has a length $D - d = (\rho/\gamma)(\beta^{-1} - 1) \approx \rho/2\gamma^3$. Thus the duration of the received pulse is $\rho/2c\gamma^3$ which means it contains frequencies up to $\omega_c \sim (c/\rho)\gamma^3$. Thus synchrotrons are a good source of X-rays.