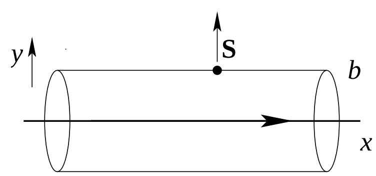
Physics 504, Lecture 22
April 18, 2011

1 Cherenkov Radiation

Before we turn to Cherenkov radiation, let's note that there is an alternate method of measuring the energy leaving the track to distances larger than the cutoff b, and that is to measure the flow of energy out of a cylinder of radius b. The energy flux through this cylinder is given by the outward component of the Poynting vector, $\vec{S} = c\vec{E}\times\vec{B}/4\pi$. As we saw by considering $\vec{E}$ at $(0, b, 0)$, there is no component of $\vec{E}$ parallel to the surface perpendicular to $\vec{v}$, so the outward flux $S_2 = -cE_1B_3/4\pi$. Thus



$$\frac{\partial E}{\partial t} = \frac{c}{4\pi}2\pi b\int_{-\infty}^{\infty} dx E_1(x, b, 0, t)B_3(x, b, 0, t).$$

The fields at any x vary with time as the particle passes, but the situation is invariant under $t \to t + \tau$, $x \to x + v\tau$. Integrating over all x at an instant is equivalent to integrating at one point in x over all time, with $dx \to v\,dt$. Also the rate of loss in x is just $1/v$ times the rate of loss in t, so

$$\begin{aligned}\frac{\partial E}{\partial x} = (1/v)\frac{\partial E}{\partial t} &= \frac{c}{4\pi}2\pi b\int_{-\infty}^{\infty} dt\, E_1(0, b, 0, t)B_3(0, b, 0, t)\\ &= cb \text{ Re } \int_0^{\infty} d\omega B_3^*(\omega)E_1(\omega)\end{aligned}$$

where the fields are evaluated at $(0, b, 0)$. From last time we have

$$E_1(\omega) = -i\sqrt{\frac{2}{\pi}}\frac{ze\omega}{v^2}\left(\frac{1}{\epsilon(\omega)} - \beta^2\right)K_0(\lambda b).$$

We also saw the source for $\vec{A}$ is $\vec{J}$, so it has only an x component, so $B_3(\vec{k}, \omega) = -ik_2A_1 = -i\epsilon(\omega)k_2(v/c)\Phi(\vec{k}, \omega) = \epsilon(\omega)(v/c)E_2(\vec{k}, \omega)$, so using the result for E_2 from last time,

$$B_3(\vec{x} = (0, b, 0), \omega) = \sqrt{\frac{2}{\pi}}\frac{ze\lambda}{c}K_1(\lambda b)$$

we have

$$\begin{aligned}\left(\frac{dE}{dx}\right) &= bc \text{ Re } \int_0^{\infty} d\omega\, E_1(\omega)B_3^*(\omega)\\ &= \frac{2}{\pi}\frac{z^2e^2}{v^2} \text{ Re } \int_0^{\infty} d\omega\, (i\omega\lambda^* b)K_1(\lambda^* b)K_0(\lambda b)\left(\frac{1}{\epsilon(\omega)} - \beta^2\right).\end{aligned}$$

This result is due to Fermi.

2 Cherenkov Radiation

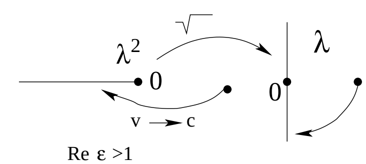
If, instead of concentrating on the energy lost past an atomic-sized cylinder by the incident particle, we ask about the energy radiated out to a macroscopic distance a, we have $\lambda a >> 1$, and we may use the asymptotic forms of the modified Bessel functions $K_\nu(z) = \sqrt{\frac{\pi}{2z}}e^{-z}$, so

$$\left(\frac{dE}{dx}\right) = \frac{2}{\pi}\frac{z^2e^2}{v^2} \text{ Re } \int_0^{\infty} d\omega\, \frac{i\omega\lambda^* a}{\sqrt{\lambda\lambda^*}}\frac{\pi}{2a}\left(\frac{1}{\epsilon(\omega)} - \beta^2\right)e^{-2 \text{ Re } \lambda a}.$$

The expression for λ comes from the appearence in the integrals over $\vec{k}$ of

$$\lambda^2 = \frac{\omega^2}{v^2}\left(1 - \beta^2\epsilon(\omega)\right),$$

which is surely meant to be mostly positive and have a positive square root for low velocities or as $\omega \to \infty$, where $\epsilon(\omega) \to 1$. Thus the energy flux reaching a large distance a vanishes exponentially as long as λ maintains its real part, which it will do as long as β^2 Re $\epsilon < 1$. λ does have a small negative imaginary part, however, because for real ω, $\epsilon(\omega)$ has a positive imaginary part. If there is a region of real ω for which β^2 Re $\epsilon(\omega) > 1$, in that region λ will be imaginary with a negative imaginary part, and the dependence on a disappears, so that the energy is escaping to infinity. For λ rotating from positive to negative imaginary, $\sqrt{\lambda^*/\lambda} \to i$, and for macroscopic a



$$\left(\frac{dE}{dx}\right) = \frac{z^2e^2}{c^2} \text{ Re } \int_{\beta^2\epsilon(\omega)>1} d\omega\,\omega\left(1 - \frac{1}{\beta^2\epsilon(\omega)}\right).$$

As there is no dimunition of the energy as b grows, we must be in the radiation region, where locally we have a wave moving in the $\vec{E}\times\vec{B}$ direction. As $\vec{A} \parallel \vec{v}$, $\vec{B}$ field is purely azimuthal, in the z direction at $\vec{x} = (0, b, 0)$, so the angle $\theta_C = \tan^{-1}(dy/dx)$ of the emitted light has $\tan\theta_C = -E_1/E_2$. In evaluating

$$E_1 = i\frac{ze\omega}{c^2}\left[1 - \frac{1}{\beta^2\epsilon(\omega)}\right]\frac{e^{-\lambda b}}{\sqrt{\lambda b}}$$

we note that as $\lambda^2 := \omega^2(1-\beta^2\epsilon(\omega)/v^2$, the term in brackets is $-v^2\lambda^2/\omega^2\beta^2\epsilon(\omega) = -c^2\lambda^2/\omega^2\epsilon(\omega)$. We also have

$$E_2 = \frac{ze}{v\epsilon(\omega)}\sqrt{\frac{\lambda}{b}}e^{-\lambda b}$$

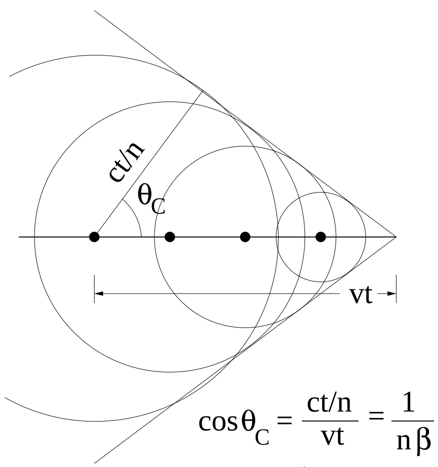
so

$$\tan\theta_C = -\frac{E_1}{E_2} = -i\frac{\omega\epsilon(\omega)v}{c^2}\frac{-c^2\lambda^2}{\omega^2\epsilon(\omega)\lambda} = i\frac{\lambda v}{\omega} = \sqrt{\beta^2\epsilon(\omega) - 1},$$

where we have made use of the fact that $\lambda = -i|\lambda|$ in the range of ω which radiates. But then

$$\cos\theta_C = \frac{1}{\sqrt{1+\tan^2\theta_C}} = \frac{1}{\beta\sqrt{\epsilon(\omega)}} = \frac{1}{\beta n(\omega)}.$$

We have needed an elaborate calculation to find the amplitude of the radiation, but the angle can be derived by freshmen, by considering the wavefront composed of spherical wavelets emitted at various times, each propagating outward with velocity $c/n = c/\sqrt{\epsilon}$. Simple geometry shows that $\cos\theta_C$ is the ratio of this speed to the speed with which the particle moves through the medium.



Note that the radiation is 100% polarized because the $\vec{E}$ field is in the plane including the charged particle's trajectory, while $\vec{B}$ is normal to that plane.

2.1 Hard Scattering

Last time when we began talking about what happens to a fast particle in media, we mentioned that two effects, the loss of energy and the scattering (change in direction) are largely due to different causes and can be treated separately. We then focussed on energy loss, due to scattering off electrons, either individual ones or those giving rise to the dielectric constant of the medium. As we mentioned, changes in direction of the particle come primarily from scattering off something as heavy or heavier, for example the nuclei of atoms, for a meson or baryon projectile. This scattering is also primarily due to Rutherford scattering, but this time the lab is nearly the center of mass frame, and the angle is primarily the angle of scattering of the projectile.

The cross section is highly dominated by small angles, so we can write

$$\frac{d\sigma}{d\Omega} = \left(\frac{2zZe^2}{pv}\right)^2\frac{1}{\theta^4}$$

for a nucleus of charge Ze, where p and v are the momentum and speed of the projectile. This formula has two limits of applicability. Integrating over $d\Omega = 2\pi\sin\theta\,d\theta$ gives infinity, while there ought to be essentially no scattering for impact parameters much greater than the size of the electron shells. That is, we considered the unscreened potential of the nucleus, but this electric field is completely screened out by the electrons if the projectile is further than the atomic size from the nucleus. A phenomenological treatment is to take

$$\frac{d\sigma}{d\Omega} = \left(\frac{2zZe^2}{pv}\right)^2\frac{1}{(\theta^2+\theta^2_{\rm min})^2}.$$

$\theta_{\rm min}$ is not really a minimum scattering angle — there is still significant scattering down to $\theta = 0$. Rather, it is a parameter about which Jackson provides several suitable choices, all roughly such that the total cross section will be the cross section of the atom. That is,

$$\begin{aligned}\sigma &= 2\pi\left(\frac{2zZe^2}{pv}\right)^2\int_0^\pi\frac{\sin\theta}{(\theta^2+\theta^2_{\rm min})^2}d\theta \approx 2\pi\left(\frac{2zZe^2}{pv}\right)^2\int_0^\infty\frac{\theta\,d\theta}{(\theta^2+\theta^2_{\rm min})^2}\\ &= \pi\left(\frac{2zZe^2}{pv}\right)^2\int_0^\infty\frac{du}{(u+\theta^2_{\rm min})^2} = \left(\frac{2zZe^2}{pv}\right)^2\frac{\pi}{\theta^2_{\rm min}}.\end{aligned}$$

If a is the rough radius of the electron shell, and we set $\sigma = \pi a^2$, we have

$$\theta_{\rm min} = \frac{2zZe^2}{pva}. \qquad \text{(classical)}$$

There is also, of course, a largest angle $\theta_{\rm max}$. Not only is $\theta \leq \pi$, but there is a cutoff that comes from the projectile penetrating the finite-sized nucleus.

The projectile will undergo a large number of scatterings through small angles. The mean change in direction will be zero, of course, but

$$\begin{aligned}
\langle\theta^2\rangle &= \frac{\int \theta^2 \sin\theta (d\sigma/d\Omega)\, d\theta}{\int \sin\theta (d\sigma/d\Omega) d\theta} \approx \frac{\int_0^{\theta_{\rm max}} \theta^3 d\theta/(\theta^2+\theta^2_{\rm min})^2}{\int_0^{\theta_{\rm max}} \theta d\theta/(\theta^2+\theta^2_{\rm min})^2} \\
&= \frac{\int_0^{\theta^2_{\rm max}} du\, u/(u+\theta^2_{\rm min})^2}{\int_0^{\theta^2_{\rm max}} du/(u+\theta^2_{\rm min})^2} \\
&= \frac{\ln(u+\theta^2_{\rm min})\Big|_0^{\theta^2_{\rm max}} + \theta^2_{\rm min}/(\theta^2_{\rm max}+\theta^2_{\rm min}) - 1]}{1/\theta^2_{\rm min} - 1/(\theta^2_{\rm max}+\theta^2_{\rm min})} \approx 2\theta^2_{\rm min} \ln \frac{\theta_{\rm max}}{\theta_{\rm min}}.
\end{aligned}$$

The number of scatterings in traversing a thickness t is $N\sigma t$, and the mean square of the independent scatterings is the sum of the individual mean squares, so if Θ is the total change in angle (in thickness t),

$$\langle\Theta^2\rangle = N\sigma t \langle\theta^2\rangle = 2\pi N \left(\frac{2zZe^2}{pv}\right)^2 \ln\left(\frac{\theta_{\rm max}}{\theta_{\rm min}}\right) t.$$

This fuzziness in the direction of the track will limit the accuracy with which one can determine the initial direction of a charged particle emerging from a collision in a detector, or determine the momentum of a charged particle from its track bending in a magnetic field.

We will skip the rest of Chapter 13.

3 Radiation by Moving Charges

We will now consider the radiation field produced by a moving charge undergoing a specified motion through space-time. We are assuming no incoming field, so the electromagnetic field is that given by the retarded Green's

function with the point particle source. In lecture 19 we found the Green's function is

$$D_r(z^\mu) = \frac{\Theta(z^0)}{4\pi R}\delta(z^0 - R), \tag{1}$$

where $R = |\vec z|$, and the source of a point particle is

$$J^\mu(x^\nu) = qc \int d\tau\, \delta^4(x^\nu - r^\nu(\tau))U^\mu(\tau), \tag{2}$$

where $r^\mu(\tau)$ is the worldline position of the particle at its proper time τ, and $U^\mu(\tau)$ is its 4-velocity at the same space-time event. The Green's function can be written more covariantly by noting that $\Theta(z^0)\delta(z^\mu z_\mu) = \Theta(z^0)\delta(z_0^2 - R^2) = \Theta(z^0)\delta[(z_0 - R)(z_0 + R)] = \frac{1}{2R}\delta(z_0 - R)$, as z_0 and R are both non-negative. So we may rewrite

$$D_r(z^\mu) = \frac{\Theta(z^0)}{2\pi}\delta(z_\mu z^\mu), \tag{3}$$

which is a manifestly covariant form[1].

The radiation field is thus

$$\begin{aligned}
A^\mu(x^\nu) &= \frac{4\pi}{c}\int d^4x' D_r(x - x')J^\mu(x') \\
&= 2q\int d^4x' d\tau \Theta(x^0 - x'^0)\delta((x-x')^2)\delta^4(x'^\nu - r^\nu(\tau))U^\mu(\tau) \\
&= 2q\int d\tau \Theta(x^0 - r^0(\tau))\delta((x - r(\tau))^2)U^\mu(\tau).
\end{aligned} \tag{4}$$

We can use the remaining δ function to do the τ integral, using $\delta(f(\tau)) = \sum_{\tau_j} \frac{1}{|df/d\tau|_{\tau_j}}\delta(\tau - \tau_j)$, where τ_j are the set of points for which $f(\tau)$ vanishes. In the current case, f vanishing means $r^\mu(\tau)$ lies on the light-cone of the point x^μ, and the Θ function restricts our attention to the single point in the past that the particle crossed this light-cone[2]. As $d(x - r(\tau))^2/d\tau = -2(x^\rho - r^\rho(\tau))U_\rho(\tau)$, we find

$$A^\mu(x^\nu) = q\left.\frac{U^\mu(\tau)}{(x^\rho - r^\rho(\tau))U_\rho}\right|_{\tau_0},$$

where the functions of τ are evaluated at the one point in the past where the particle left the lightcone of x^ν. This 4-vector potential is known as the Liénard-Wiechert potential.

[1] Under proper isochronous Lorentz transformation. Obviously not under time reversal.

[2] We assume the particle has a mass and so is always travelling at a velocity less than c, and that it is not passing exactly through the point x^μ.

To evaluate the electric and magnetic fields, or $F^{\mu\nu}$, we apply a derivative to (4):

$$\begin{aligned}\partial^\alpha A^\beta &= 2q\int d\tau\Big[\left(\partial^\alpha\Theta(x^0-r^0(\tau))\right)\delta((x-r(\tau))^2)U^\beta(\tau)\\ &\qquad +\Theta(x^0-r^0(\tau))\partial^\alpha\delta((x-r(\tau))^2)U^\beta(\tau)\Big].\end{aligned}$$

The first term contains $\partial^\alpha\Theta(x^0-r^0(\tau))=\delta^\alpha_0\delta(x^0-r^0(\tau))$, which contributes only if x^μ and $r^\mu(\tau)$ are at the same time. But this is multiplied by a δ function that requires $r^\mu(\tau)$ to be on the light-cone of x^μ, which rules out all such points except the point $x^\mu=r^\mu(\tau)$, and we have already assumed the particle does not pass exactly through the point in question. This leaves the term with $\partial^\alpha\delta(f(x^\mu,\tau))$, where $f=(x^\mu-r^\mu(\tau))^2$. As the delta function only depends on f, the chain rule says

$$\begin{aligned}\partial^\alpha\delta(f(x^\mu,\tau)) &= \left[\frac{d}{df}\delta(f)\right]\partial^\alpha f=\left[\left(\frac{df}{d\tau}\right)^{-1}\frac{d}{d\tau}\delta(f)\right]2(x^\alpha-r^\alpha(\tau))\\ &= -\frac{(x-r(\tau))^\alpha}{(x-r(\tau))_\rho U^\rho}\frac{d}{d\tau}\delta(f).\end{aligned}$$

Then

$$\begin{aligned}\partial^\alpha A^\beta &= -2q\int d\tau\Theta(x^0-r^0(\tau))U^\beta(\tau)\frac{(x-r(\tau))^\alpha}{(x-r(\tau))_\rho U^\rho}\frac{d}{d\tau}\delta((x^\mu-r^\mu(\tau))^2)\\ &= 2q\int d\tau\Theta(x^0-r^0(\tau))\delta((x^\mu-r^\mu(\tau))^2)\frac{d}{d\tau}\left(\frac{U^\beta(\tau)(x-r(\tau))^\alpha}{(x-r(\tau))_\rho U^\rho}\right),\end{aligned}$$

where we have integrated by parts, discarding the $d\Theta/d\tau$ term as before, and discarding surface terms as the particle was not on the light-cone at infinity. As we did for A^μ, the remaining delta function can be used to do the τ integral, which gives another factor of $U\cdot(x-r)$ in the denominator, and

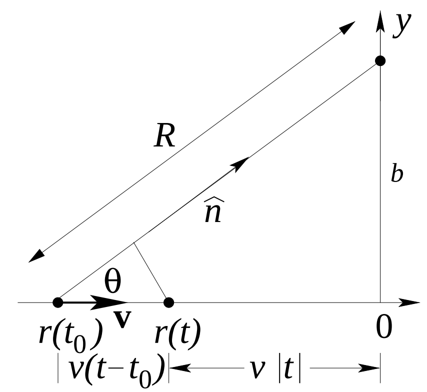
$$F^{\alpha\beta}=\frac{q}{U_\rho(x^\rho-r^\rho(\tau))}\frac{d}{d\tau}\left[\frac{(x-r(\tau))^\alpha U^\beta(\tau)-(x-r(\tau))^\beta U^\alpha(\tau)}{U_\mu(x^\mu-r^\mu(\tau))}\right]\Bigg|_{\tau_0}. \qquad (5)$$

The τ derivative acts on an expression involving the position and the velocity of the charged particle. When it acts on the velocities, it gives a piece proportional to the acceleration of the particle. When the derivative acts on r, it either kills one of the factors of $x-r$ in the numerator or squares one in the denominator, so these terms fall off as $\frac{1}{|\vec{x}|^2}$, and represent static fields. The acceleration piece, however, behaves at large distances as we expect for radiation, falling as $1/|\vec{x}|$ and having transverse $\vec{E}$ and $\vec{B}$.

Consider the effect of a uniformly moving charge, without acceleration. Then the derivative cannot act on a U and must act on one factor of $x^\sigma-r^\sigma(\tau)$, on which it gives $-U^\sigma$. The terms which come from differentiating the numerator cancel, and the derivative of the denominator is just $-U^2=-c^2$, so we have

$$F^{\alpha\beta}=\frac{qc^2}{(U_\rho(x^\rho-r^\rho(\tau)))^3}\left[(x-r(\tau))^\alpha U^\beta(\tau)-(x-r(\tau))^\beta U^\alpha(\tau)\right].$$

Let the particle be moving along the x axis, and let us observe from the point $x=z=0$, $y=b$. Set our clock to zero at the moment the particle crosses $x=0$. Then we have $\vec{r}=vt\hat{e}_x$, for which $U^\alpha=(\gamma c,\gamma v,0,0)$ and $r^\alpha(\tau)=U^\alpha\tau$, and we are observing at the point $x^\mu=(ct,0,b,0)$. The particle left the light-cone at time t_0 for which $(x^\mu-r^\mu(t_0))^2=0$. $x^\mu-r^\mu(t_0)=(c(t-t_0),-vt_0,b,0)$, so $c^2(t-t_0)^2-v^2t_0^2-b^2=0$. or $t_0=\gamma^2\left(t-\sqrt{t^2\beta^2+b^2/c^2\gamma^2}\right)$.



The diagram shows the position of the particle at the time t_0 at which it can emit the lightlike ray towards the observer a distance R away. Let $\hat{n}$ be a unit 3-vector in that direction. As drawn t is negative, and the distance already travelled ($v(t-t_0)$) and the distance yet to be travelled in time $-t$ are indicated. The quantity $U_\alpha(x^\alpha-r^\alpha(t_0))=\gamma(c^2(t-t_0)+v^2t_0)=c^2\gamma(t-\gamma^{-2}t_0)=c^2\gamma\sqrt{t^2\beta^2+b^2/c^2\gamma^2}=c\sqrt{b^2+v^2\gamma^2t^2}$.

Let us evaluate the y component of the electric field:

$$E_2=F_{02}=qc^2\frac{(x-r)_2U_0}{(U_\alpha(x-r)^\alpha)^3}=\frac{qb\gamma}{(b^2+v^2\gamma^2t^2)^{3/2}}.$$

4 Power Radiated

Our previous calculation of a nonaccelerating particle had no radiation, and the results could have been found by transforming to the rest frame and using Coulomb's law. More interesting is what happens if the particle accelerates. The power radiated per unit area is given by the Poynting vector

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} \rightarrow \frac{c}{4\pi}\vec{E}^{\,2}\hat{n}.$$

We could evaluate this directly from our complete formula (5), but Jackson pursues a clever trick — the instantaneous power can be evaluated in the momentary rest frame. As energy and time transform the same way, the total power emitted is an invariant. In the momentary rest frame $U^\alpha = (c, 0, 0, 0)$, $dU^\alpha/d\tau = (0, \dot{\vec{v}})$, and if we assume we are observing from $\vec{R}$ at time t, and the particle was at $\vec{x} = 0$ at the time $t_0 = t - R/c$ that it crossed out of our past light-cone, $r^\mu(\tau_0) = (ct - R, \vec{0})$, $x^\mu - r^\mu = (R, \vec{R}) = R(1, \hat{n})$, $U\cdot(x - r) = Rc$

$$\begin{aligned}
\vec{E} = \sum_i F_{0i}\hat{e}_i &= \frac{q}{Rc}\left[\frac{R(-\dot{\vec{v}}) - \vec{R}\cdot 0}{cR} - \frac{R\cdot 0 - c\vec{R}\,(-\dot{\vec{v}})\cdot\vec{R}}{c^2R^2}\right] \\
&= -\frac{q}{c^2R}\left[\dot{\vec{v}} + \hat{n}\,\hat{n}\cdot\dot{\vec{v}}\right] \\
&= \frac{q}{c^2R}\hat{n}\times(\hat{n}\times\dot{\vec{v}}).
\end{aligned}$$

Then the power per sterradian is

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}|\hat{n}\times\dot{\vec{v}}|^2 = \frac{q^2}{4\pi c^3}|\dot{\vec{v}}\,|^2\sin^2(\psi),$$

where ψ is the angle between the acceleration and the vector $\hat{n}$ pointing to the observer. The integral gives

$$P = \frac{2q^2}{3c^3}|\dot{\vec{v}}\,|^2.$$

Jackson argues that we can get the relativistic equation be noting that the power needs to be an invariant expression built from U^α (or p^α) and the first derivative $dp^\alpha/d\tau$. The formula in the rest frame can be expressed as

$$P = \frac{2}{3}\frac{q^2}{m^2c^3}\frac{d\vec{p}}{dt}\cdot\frac{d\vec{p}}{dt} = -\frac{2}{3}\frac{q^2}{m^2c^3}\frac{dp^\alpha}{d\tau}\frac{dp_\alpha}{d\tau} \qquad \text{in the rest frame,}$$

but the last expression is invariant. In any other frame, it gives

$$P = \frac{2}{3}\frac{q^2}{m^2c^3}\left[\left(\frac{d\vec{p}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2\right].$$

Using $E = mc^2\gamma$, $\vec{p} = mc\gamma\vec{\beta}$, and $d/d\tau = \gamma d/dt$, and noting from $\gamma^{-2} = 1 - \beta^2 \Longrightarrow -2\gamma^{-3}d\gamma = -2\beta d\beta \Longrightarrow d\gamma = \gamma^3\beta d\beta$, the term in brackets is m^2c^2 times

$$\begin{aligned}
&\gamma^2\left[\left(\gamma^3\beta\dot{\beta}\vec{\beta} + \gamma\dot{\vec{\beta}}\right)^2 - (\gamma^3\beta\dot{\beta})^2\right] \\
&= \gamma^2\left[\gamma^6\beta^4(\dot{\beta})^2 + 2\gamma^4\beta\dot{\beta}\vec{\beta}\cdot\dot{\vec{\beta}} + \gamma^2(\dot{\vec{\beta}})^2 - \gamma^6\beta^2\dot{\beta}^2\right] \\
&= \gamma^6\dot{\beta}^2\left(\gamma^2\beta^4 - \gamma^2\beta^2 + 2\beta^2)\right) + \gamma^4(\dot{\vec{\beta}})^2
\end{aligned}$$

because $\vec{\beta}\cdot\dot{\vec{\beta}} = \frac{1}{2}d\vec{\beta}^2/dt = \frac{1}{2}d\beta^2/dt = \beta\dot{\beta}$. But $\gamma^2(\beta^4 - \beta^2) = -\beta^2$, so

$$P = \frac{2}{3}\frac{q^2}{c}\gamma^6\left(\gamma^{-2}(\dot{\vec{\beta}})^2 - \beta^2\dot{\beta}^2\right).$$

The parentheses may be rewritten $(\dot{\vec{\beta}})^2 - \beta^2\left((\dot{\vec{\beta}})^2 - \dot{\beta}^2\right) = (\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2$ because $(\vec{\beta}\times\dot{\vec{\beta}})^2 = (\vec{\beta})^2(\dot{\vec{\beta}}) - (\vec{\beta}\cdot\dot{\vec{\beta}})^2$ and the last term is $-\beta^2\dot{\beta}^2$ as explained above. So all in all,

$$P = \frac{2}{3}\frac{q^2}{c}\gamma^6\left[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right].$$

The rest of section 14.2 is certainly important but straightforward, so I will not rewrite it. You should read it.