Physics 504, Lecture 19
April 7, 2011

1 $\mathcal{L}$, $\mathcal{H}$, canonical momenta, and $T^{\mu\nu}$ for E&M

We have seen field theory needs the Lagrangian *density*

$$\mathcal{L}(A^\mu, \partial A^\mu/\partial x^\nu, J^\rho, x^\xi)$$

and the equations of motion come from the functional derivatives $\dfrac{\delta L}{\delta A^\mu(x^\nu)}$ and $\dfrac{\delta L}{\delta(\partial_\rho A^\mu(x^\nu))}$. The first is an integral over d^3x' of $\dfrac{\delta \mathcal{L}(x'^\mu)}{\delta A^\mu(x^\nu)}$, which contains a $\delta(x'-x)$ function. Because we are always integrating over x', we used partial derivative notation amd treated the $A^\mu(x^\nu)$ dependence of $\mathcal{L}$ as if it were a simple argument instead of a function. This gave us equations of motion at each point in space-time. These Euler-Lagrange equations involved not a total momentum but a momentum *density*. For a scalar field ϕ_j the *canonical momentum density* is the $\mu = 0$ component of the momentum 4-vector current

$$\Pi_j^\mu(x^\rho) = \left.\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_j}\right|_{x^\rho}.$$

As we have not a scalar but four fields A^ν, we have four 4-vector fields

$$\Pi_\alpha{}^\mu := \frac{\partial \mathcal{L}}{\partial\left(\dfrac{\partial A^\alpha(\vec{x},t)}{\partial x^\mu}\right)}.$$

Last time we saw that the lagrangian *density* for the electromagnetic fields is

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J_\mu A^\mu,$$

so the canonical momentum currents are

$$\Pi_\alpha{}^\mu := \frac{\partial \mathcal{L}}{\partial(\partial A^\alpha(\vec{x},t)/\partial x^\mu)} = -\frac{1}{4\pi} F^\mu{}_\alpha,$$

because, as we saw last time, this only involves the F^2 term.

1.1 The Stress (Energy-Momentum) Tensor

In discrete mechanics we define the Hamiltonian by $H = \sum_i P_i \dot{q}_i - L$ and then substitute for $\dot{q}_i$ the expression for it in terms of P_j. In field theory we start with the Lagrangian density, and the fields $P_i(\vec{x})$ and $\phi(\vec{x})$, so we get the Hamiltonian density

$$\mathcal{H}(\vec{x}) := \sum_i P_i(\vec{x})\dot{\phi}_i(\vec{x}) - \mathcal{L}(\vec{x}) = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial\phi_i/\partial x^0)} \frac{\partial \phi_i}{\partial x^0} - \mathcal{L}.$$

We see that this is naturally defined with two Lorentz indices, both of which are 0, so it is one component of a tensor

$$T^\mu{}_\nu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial\phi_i/\partial x^\mu)} \frac{\partial \phi_i}{\partial x^\nu} - \delta^\mu_\nu \mathcal{L}.$$

This object goes by the names **energy-momentum tensor** or **stress-energy tensor** or **canonical stress tensor**. For electromagnetism, ϕ_i is replaced by A^λ, the first factor in the first term is

$$\frac{\partial \mathcal{L}}{\partial(\partial A^\lambda/\partial x^\mu)} = -\frac{1}{4\pi} F^\mu{}_\lambda,$$

the second factor is

$$\frac{\partial A^\lambda}{\partial x^\nu},$$

so our first (tentative) expression for the energy momentum tensor is

$$T_{\mu\nu} = -\frac{1}{4\pi}\left(F_{\mu\lambda}\frac{\partial A^\lambda}{\partial x^\nu} - \frac{1}{4}\eta_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right).$$

This tensor has some good properties and some bad properties. We have seen that its 00 component is the hamiltonian density, which we may interpret as the energy density. We might expect, then, that T^{0i} could be interpreted as the density of momentum, and the integral of it, over all space, the i'th component of the total momentum. But we see that

$$T^{0i} = \frac{1}{4\pi}F_{0\lambda}\partial_i A^\lambda = \frac{1}{4\pi}E_j \partial_i A_j = \frac{1}{4\pi}\left(\vec{E}\times\vec{B} + \left(\vec{E}\cdot\vec{\nabla}\right)\vec{A}\right)_i.$$

We expect the $\vec{E}\times\vec{B}$ term from the expression for the Poynting vector, but not the last term, which is not even gauge invariant. It is, however, in the absence of charges (and we have not included the momentum of any charges) a total derivative (as then $\vec{\nabla} \cdot \vec{E} = 0$, and $\vec{\nabla} \cdot (A_i \vec{E}) = \left(\vec{E} \cdot \vec{\nabla}\right) \vec{A}_i + A_i \vec{\nabla} \cdot \vec{E}$). So the integral of this density will give what we want. Thus we have the good property that

$$\int d^3x T^{0\mu} = P^\mu, \qquad \text{the total momentum.} \tag{1}$$

The way $T^{\mu\nu}$ is defined in general guarantees that, if the Lagrangian has no *explicit* dependence on x^μ, the (stream) divergence $\partial_\mu T^\mu{}_\nu$ will vanish when evaluated on fields which obey the equations of motion. We have

$$\partial_\mu T^\mu{}_\nu = \sum_i \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \partial_\nu \phi_i + \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu \partial_\nu \phi_i - \partial_\nu \mathcal{L}.$$

The derivative in the last term is given by the chain rule

$$-\partial_\nu \mathcal{L} = -\sum_i \frac{\partial \mathcal{L}}{\partial \phi_i} \partial_\nu \phi_i - \sum_i \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi_i\right)} \partial_\nu \partial_\mu \phi_i$$

so

$$\partial_\mu T^\mu{}_\nu = \sum_i \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} - \frac{\partial \mathcal{L}}{\partial \phi_i} \right) \partial_\nu \phi_i$$

and the parenthesis vanishes by the equations of motion. Thus we have

$$\partial_\mu T^\mu{}_\nu = 0. \tag{2}$$

Notice that the tensor we have defined so far, $T^{\mu\nu}$ is not symmetric under $\mu \leftrightarrow \nu$, which is a problem when it comes to defining the angular momentum. If T^{0j} is truly the momentum density, we would expect $\int \epsilon_{ijk} x_j T^{0k} d^3x$ to be the angular momentum, but in fact that only works if $T^{\mu\nu}$ is symmetric.

So we have good properties (1) and (2), but we have the problems that $T^{\mu\nu}$ is not gauge invariant, is not symmetric, and differs from Poynting locally. Can we add something to T which will fix these problems without messing up the good properties? Note that if we add $\partial_\rho \psi^{\rho\mu\nu}$ to $T^{\mu\nu}$, with the requirement that $\psi^{\rho\mu\nu} = -\psi^{\mu\rho\nu}$, the extra piece will change (2) by $\partial_\mu \partial_\rho \psi^{\rho\mu\nu} = 0$, *i. e.* unchanged, because the derivatives are symmetric while ψ is antisymmetric. Furthermore, the integral over space of $T^{0\mu}$ gets an addition of $\int d^3x \partial_\rho \psi^{\rho 0\nu} = \int d^3x \partial_j \psi^{j0\nu} = \int_S n_j \psi^{j0\nu} \to 0$ where the surface S goes to infinity, where we can assume all our fields go to zero[1]. Thus adding $\partial_\rho \psi^{\rho\mu\nu}$ preserves all the good properties.

So consider $\psi^{\rho\mu\nu} = A^\nu F^{\mu\rho}/4\pi$, and adding

$$\frac{1}{4\pi}\partial_\rho \left(A^\nu F^{\mu\rho}\right) = \frac{1}{4\pi}\left(\partial_\rho A^\nu\right) F^{\mu\rho}$$

because $\partial_\rho F^{\mu\rho} = 0$ in the absence of a source J^μ. But this is just what we need to add to $T^{\mu\nu}$ to make

$$\Theta^{\mu\nu} = T^{\mu\nu} + \frac{1}{4\pi}F^{\mu\rho}\partial_\rho A^\nu = -\frac{1}{4\pi}\left(F^{\mu\rho}F^\nu{}_\rho - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right).$$

This expression has all the good properties and is also gauge invariant and symmetric. Furthermore,

$$\Theta^{0i} = -\frac{1}{4\pi}F^{0j}F^i{}_j = \frac{1}{4\pi}E_j\epsilon_{ijk}B_k = \frac{1}{4\pi}(\vec{E}\times\vec{B})_i,$$

the correct momentum density or energy flux, as given by Poynting.

1.2 Ambiguities in the Action

The action we used for the electromagnetic field by itself depends only on $F_{\mu\nu}$, that is, on the electric and magnetic fields, but the interaction term with currents, $A_{\text{int}} = -(1/c)\int d^4x J_\mu A^\mu$ depends on the 4-vector potential, which, as we know, is not uniquely defined, because the physical fields $\vec{E}$ and $\vec{B}$ are unchanged by a gauge transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu\Lambda$, that is, we could add a piece to the action of $-(1/c)\int d^4x J^\mu \partial_\mu \Lambda$. This would, however, make no difference to the equations of motion, because the

$$\int d^4x J^\mu \partial_\mu \Lambda = \int_S n_\mu J^\mu \Lambda - \int d^4x \Lambda \partial_\mu J^\mu,$$

where S is a hypersurface surrounding the four dimensional region we are considering, which means a hypersurface at infinity. We can provide some excuse for claiming either J^μ or Λ vanishes at infinity, so the (hyper)surface term can be discarded, and the other term involves the divergence of the current, which we know has to be zero by conservation of charge.

[1]We often take a cavalier attitude about such arguments, but you should keep a small reservation in the back of your head that under some circumstances there may be anomolies that make it impossible to assure that these terms can be ignored

This invariance is a general feature of lagrangian mechanics. Because it is only the variation of the lagrangian, and not the lagrangian, that matters physically, and the fields are varied only inside the region and not on the surface, any change in the lagrangian density by a divergence, or of the lagrangian by a total time derivative, is irrelevant to the physics.

1.3 $\Theta^{\mu\nu}$ in the presence of currents

We saw that the energy-momentum tensor of the electromagnetic field can be considered to be

$$\Theta^{\mu\nu} = -\frac{1}{4\pi}\left(F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right),$$

and in the absence of any sources, it is conserved, $\partial_\mu\Theta^{\mu\nu} = 0$. What happens if there are sources? Now

$$\begin{aligned}
4\pi\partial_\mu\Theta^{\mu\nu} &= \partial_\mu\left(F^{\mu\rho}F_\rho{}^\nu + \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) \\
&= \left(\partial_\mu F^{\mu\rho}\right)F_\rho{}^\nu + F^{\mu\rho}\partial_\mu F_\rho{}^\nu + \frac{1}{2}F^{\alpha\beta}\partial^\nu F_{\alpha\beta} \\
&= \frac{4\pi}{c}J^\rho F_\rho{}^\nu + \frac{1}{2}F^{\alpha\beta}\left(\partial_\alpha F_\beta{}^\nu - \partial_\beta F_\alpha{}^\nu + \partial^\nu F_{\alpha\beta}\right) \\
&= \frac{4\pi}{c}J^\rho F_\rho{}^\nu + \frac{1}{2}F^{\alpha\beta}\eta^{\nu\rho}\left(\partial_\alpha F_{\beta\rho} + \partial_\beta F_{\rho\alpha} + \partial_\rho F_{\alpha\beta}\right) \\
&= \frac{4\pi}{c}J^\rho F_\rho{}^\nu,
\end{aligned}$$

as the term in parentheses is zero by the homogeneous Maxwell equations.

Thus the total 4-momentum of the electromagnetic field

$$P^\nu_{\text{EM}} = \frac{1}{c}\int d^3x\Theta^{0\nu}(\vec{x}),$$

is not conserved, but rather

$$\begin{aligned}
\frac{dP^\nu_{\text{EM}}}{dt} &= \frac{1}{c}\frac{d}{dt}\int d^3x\,\Theta^{0\nu}(\vec{x}) = \int d^3x\,\partial_0\Theta^{0\nu}(\vec{x}) \\
&= \frac{1}{c}\int d^3x J^\rho(\vec{x})F_\rho{}^\nu(\vec{x}) - \frac{1}{c}\int d^3x\partial_i\Theta^{i\nu} = \frac{1}{c}\int d^3x J^\rho(\vec{x})F_\rho{}^\nu(\vec{x}),
\end{aligned}$$

as the integral of a divergence can be thrown away as a surface term at infinity.

Consider a charged particle of mass m_i, charge q_i at point $\vec{x}_i(t)$. Its mechanical 4-momentum changes by

$$\frac{dP^\nu_{(i)}}{dt} = \frac{1}{\gamma_i}\frac{dP^\nu_{(i)}}{d\tau} = \frac{1}{\gamma_i}\frac{q_i}{c}F^\nu{}_\rho(\vec{x}_i)U_i^\rho.$$

This particle corresponds to a 4-current

$$J^\rho = (c\rho, \vec{J}) = (cq_i\delta^3(\vec{x}-\vec{x}_i), q_iu_i\delta^3(\vec{x}-\vec{x}_i) = q_i\gamma_i^{-1}U_i^\rho\delta^3(\vec{x}-\vec{x}_i).$$

Plugging this into our expression for the change in the momentum of the electromagnetic field, we have

$$\frac{dP^\nu_{\text{EM}}}{dt} = \frac{q_i}{c}\int d^3x F_\rho{}^\nu(\vec{x})\gamma_i^{-1}U_i^\rho\delta^3(\vec{x}-\vec{x}_i) = -\frac{q_i}{c\gamma_i}F^\nu{}_\rho(\vec{x}_i)U_i^\rho,$$

and the *total momentum*, $P^\nu_{\text{EM}} + P^\nu_{(i)}$ is conserved.

1.4 Equation of Motion for A^μ

We saw that the equations of motion for the 4-vector potential are

$$\partial_\sigma F^{\sigma\mu} = \partial_\sigma\partial^\sigma A^\mu - \partial^\mu\partial_\sigma A^\sigma = \frac{4\pi}{c}J^\mu.$$

If we had an equation that told us to inforce the Lorenz condition $\partial_\sigma A^\sigma = 0$, we could drop the second term and have the equation

$$\partial_\sigma\partial^\sigma A^\mu = \frac{4\pi}{c}J^\mu,$$

which has as its solutions a particular solution given in terms of the Green's function for the wave equation, together with an arbitrary solution of the homogeneous equation $\partial_\sigma\partial^\sigma A^\mu = 0$. Let us discuss this homogeneous solution first. With the Lorenz condition imposed, the solution is simply

$$\sum_{\vec{k}}\left(A^\mu_{\vec{k}+}e^{i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}t} + A^\mu_{\vec{k}-}e^{i\vec{k}\cdot\vec{x}+i\omega_{\vec{k}}t}\right),$$

where $\omega = c|\vec{k}|$. This solution is constrained by the Lorenz condition to have $\omega A^0_{\vec{k}\pm} \mp \vec{k}\cdot\vec{A}_{\vec{k}\pm} = 0$. These are the solutions for an electromagnetic wave in empty space.

But if we don't impose the ad-hoc Lorenz condition, the equations

$$\partial_\sigma \partial^\sigma A^\mu - \partial^\mu \partial_\sigma A^\sigma = 0$$

are not enough to determine the evolution of $A^\mu(\vec{x}, t)$ as a function of time, even if we completely specify initial conditions on $A^\mu(\vec{x}, 0)$ and its time derivative at $t = 0$. This is most easily seen in the Fourier transformed equation

$$k_\sigma k^\sigma A^\mu - k^\mu k_\sigma A^\sigma = 0,$$

which, though it looks like four equations for A^ρ, is actually only three, because if we contract with k_μ we get

$$k_\sigma k^\sigma k_\mu A^\mu - k_\mu k^\mu k_\sigma A^\sigma = (k^2 - k^2) k_\rho A^\rho = 0,$$

which is not a constraint on A^ρ but an identity. In other words, the equation only determines the components of A transverse to k.

This is yet another indication of the gauge invariance, the statement that a gauge-invariant action principle cannot determine the evolution of a gauge-variant field because no equation will determine the gauge transformation Λ.

We can, however, adopt the Lorenz gauge condition and ask what the equation that determines A^μ in that gauge is.

So we turn to the inhomogeneous equation

$$\Box A^\mu = \partial_\beta \partial^\beta A^\mu = \frac{4\pi}{c} J^\mu,$$

with the solution

$$A^\mu(x) = \frac{4\pi}{c} \int d^4x' D(x, x') J^\mu(x'),$$

where $D(x, x')$ is a Green's function for D'Alembert's equation

$$\Box_x D(x, x') = \delta^4(x - x').$$

We are interested in solving this in all of spacetime, without boundaries at finite distances, so the equation is translation invariant and D must be a function only of the difference, $D(x, x') = D(x - x') = D(z)$. The equation may be solved by Fourier transform,

$$D(z) = \frac{1}{(2\pi)^4} \int d^4k^\mu \tilde{D}(k^\mu) e^{-ik_\mu z^\mu}.$$

As $\delta^4(z^\mu) = \frac{1}{(2\pi)^4} \int d^4k e^{-ik_\mu z^\mu}$, the solution for the Green's function is

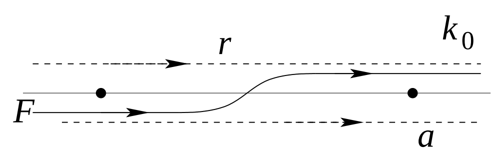
$$\tilde{D}(k^\mu) = -\frac{1}{k^2}, \quad \text{and} \quad D(z^\mu) = -\frac{1}{(2\pi)^4} \int d^4k \frac{e^{-ik_\mu z^\mu}}{k^2}.$$

Now this looks very similar to the Green's function for the Laplace equation, except that the $1/k^2$ is much more dangerous here, as it vanishes whenever $k_0^2 = \vec{k}^2$, and not just at one point in a three dimensional space. For Laplace's equation the ill-defined point in the integration was just a sign that a potential satisfying Laplace's equation could have an arbitrary constant and first derivative, the solutions of the homogeneous equation. Here too the ill-determined part of D represents the homogeneous solutions, but this is now an infinite dimensional space of solutions, all free electro magnetic waves.

The ill-defined integral through the singular point can be clarified by writing the Green's function first as

$$D(z) = -\frac{1}{(2\pi)^4} \int d^3k e^{i\vec{k}\cdot\vec{z}} \int_\Gamma dk_0 \frac{e^{-ik_0 z^0}}{k_0^2 - |\vec{k}|^2}.$$

We may make a well defined Green's function by specifying that the contour Γ should not go right along the real axis of k_0, but rather around the poles at $k_0 = \pm|\vec{k}|$. Three such contours are shown. As the integrand is analytic except at the points $k_0 = \pm|\vec{k}|$, the contours may be deformed so that they become the real k_0 axis beyond the region with the poles.



The retarded (r), advanced (a), and Feynman (F) contours for defining the Green's function.

Consider first the Green's function as given by the contour r. If the source acts at time 0, and if we evaluate $D(z)$ at a time after that, with $z^0 > 0$, the contour Γ may be closed by taking a large semicircle in the lower half complex plane, where $\left|e^{-ik_0 z^0}\right| = e^{-|\text{ Im } k_0|z^0} \underset{|k|\to\infty}{\longrightarrow} 0$, so this semicircle makes no contribution to the integral but does allow us to evaluate it as $-2\pi i$ times the sum of the two residues. The minus is due to our circling these residues clockwise rather than counterclockwise, and the residues are

$$\underset{k_0=|\vec{k}|}{\text{Res}} \frac{e^{-ik_0 z^0}}{(k_0 + |\vec{k}|)(k_0 - |\vec{k}|)} + \underset{k_0=-|\vec{k}|}{\text{Res}} \frac{e^{-ik_0 z^0}}{(k_0 + |\vec{k}|)(k_0 - |\vec{k}|)}$$

$$= \frac{e^{-i|\vec{k}|z^0}}{2|\vec{k}|} + \frac{e^{i|\vec{k}|z^0}}{-2|\vec{k}|} = -i\frac{\sin(|\vec{k}|z^0)}{|\vec{k}|}.$$

On the other hand, if $z^0 < 0$ we may close the contour in the upper half plane, as the semicircle contribution now vanishes there, and we have encircled no singularities and the Cauchy-Goursat theorem tells us it vanishes. Thus

$$D_r(z) = \frac{\Theta(z^0)}{(2\pi)^3} \int d^3k e^{i\vec{k}\cdot\vec{z}} \frac{\sin(|\vec{k}|z^0)}{|\vec{k}|}.$$

Choosing the North pole along $\vec{z}$ using spherical coordinates, this becomes

$$\begin{aligned} D_r(z) &= \frac{\Theta(z^0)}{(2\pi)^2} \int_0^\infty k^2\, dk\, d\theta \sin\theta\, e^{ikR\cos\theta}\, \frac{\sin(kz^0)}{k} \\ &= \frac{\Theta(z^0)}{2\pi^2 R} \int_0^\infty dk \sin(kR)\sin(kz^0), \end{aligned}$$

where $R = |\vec{z}|$. The Green's function is called the *retarded Green's function* because the effects occur only after the source acts. It is also called the *causal* Green's function because this is how things ought to be, though it would be perfectly consistent to use the contour a and the advanced Green's function to ask what configuration of incoming waves could be magically made to disappear by interacting with a given source J^μ.

The third contour shown in the figure gives the Feynman propagator, which is used in quantum field theory. But we need not discuss that here.

The expression for $D_r(z)$ can be further simplified by writing

$$\begin{aligned} \sin(kR)\sin(kz^0) &= \frac{1}{2}\left[\cos(k(R-z^0)) - \cos(k(R+z_0))\right] \\ &= \frac{1}{4}\left[e^{i(z_0-R)k} - e^{i(z_0+R)k} + e^{i(z_0-R)(-k)} + e^{i(z_0-R)(-k)}\right] \end{aligned}$$

so

$$\begin{aligned} D_r(z) &= \frac{\Theta(z^0)}{8\pi^2 R} \int_{-\infty}^\infty dk \left[e^{i(z_0-R)k} - e^{i(z_0+R)k}\right] \\ &= \frac{\Theta(z^0)}{4\pi R} \left[\delta(z_0-R) - \delta(z_0+R)\right] \\ &= \frac{\Theta(z^0)}{4\pi R}\delta(z_0-R), \end{aligned}$$

where the second δ was dropped because both z^0 and R are positive. So the Green's function only contributes when the source and effect are separated by a lightlike path, with $\Delta z^0 = |\Delta \vec{z}|$.

So how do we describe the field when we know what the sources are throughout space-time? We can use any of the Green's functions to get the inhomogeneous contribution, and then allow for an arbitrary solution of the homogeneous equation. Thus we can write

$$\begin{aligned} A^\mu &= A^\mu_{\rm in}(x) + \frac{4\pi}{c}\int d^4x' D_r(x-x')J^\mu(x') \\ &= A^\mu_{\rm out}(x) + \frac{4\pi}{c}\int d^4x' D_a(x-x')J^\mu(x'). \end{aligned}$$

If the sources are confined to some finite region of space-time, there will be no contribution from D_r at times earlier than the first source, and $A^\mu_{\rm in}(x)$ describes the fields before that time. Also after the last time that the source influences things, the field will be given by $A^\mu_{\rm out}(x)$ alone. Of course the source may be persistent, for example if there is a net charge, but we may often consider that the effect of the source is confined to the change from $A^\mu_{\rm in}(x)$ to $A^\mu_{\rm out}(x)$ and define the radiation field to be

$$A^\mu_{\rm rad}(x) = A^\mu_{\rm out}(x) - A^\mu_{\rm in}(x) = \frac{4\pi}{c}\int d^4x' D(x-x')J^\mu(x'),$$

where $D(z) := D_r(z) - D_a(z)$.

The expression we wrote earlier for the current density of a point charge,

$$J^\rho = q_i \gamma_i^{-1} U_i^\rho \delta^3(\vec{x} - \vec{x}_i)$$

can be written in this four-dimensional language as

$$J^\rho(x^\mu) = q_i \int dt \delta(t - x^0/c)\gamma_i^{-1} U_i^\rho \delta^3(\vec{x} - \vec{x}_i(t)) = q_i c \int d\tau \delta^4(x^\mu - x_i^\mu(\tau)) U_i^\rho,$$

where τ measures proper time along the path of the particle.