Physics 504, Lecture 11
Feb. 28, 2011

1 Radiation by Sources

We turn our attention to radiation into empty space (no waveguides) by specified sources. Again our equations are linear and time independent, so we assume all fields and sources have a time dependence $e^{-i\omega t}$. This fourier component of the electromagnetic fields will be determined by the same fourier component of the charge density $\rho(\vec{x},t) = \rho(\vec{x})e^{-i\omega t}$ and current density $\vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$. The electromagnetic fields can be specied by the scalar and vector potentials, but we recall that the scalar and vector potential have a gauge invariance In *Lorenz gauge*, where we require $\vec{\nabla}\cdot\vec{A} + \partial\Phi/c^2\partial t = 0$, we have

$$\begin{aligned}\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} &= -\rho/\epsilon_0,\\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\mu_0\vec{J}.\end{aligned}$$

Inserting the assumed time dependence, we have that both $\Phi(\vec{x})$ and each component of $\vec{A}(\vec{x})$ satisfy the inhomogeneous Helmholtz equation

$$\left(\nabla^2 + k^2\right)\Psi(\vec{x}) = -s(\vec{x}), \tag{1}$$

with $k = \omega/c$. The Green's function equation for this Helmholtz equation was derived in §6.4. I understand we may need to review this.

This equation is (inhomogeneously) linear in Ψ and is an elliptic partial differential equation. A solution is a superposition of solutions of the homogeneous equation with specific solutions giving the inhomogeneous terms, which can be built up piece by piece. We may think of the right hand side as a superposition of delta functions, $s(\vec{x}) = \int d\vec{x}\,'s(\vec{x}\,')\delta^3(\vec{x}\,'-\vec{x})$, so if we have a solution of

$$\left(\nabla_x^2 + k^2\right)G(\vec{x},\vec{x}\,') = -\delta^3(\vec{x}-\vec{x}\,') \tag{2}$$

the solution of (1) is

$$\Psi(\vec{x}) = \int d^3\vec{x}\,'\, s(\vec{x}\,')G(\vec{x},\vec{x}\,').$$

From electrostatics, we know that a point charge at the origin, with potential $V(\vec{x}) = \dfrac{q}{4\pi\epsilon_0 |x|}$ has an electric field $\vec{E} = -\vec{\nabla} V = q\dfrac{\vec{x}}{4\pi\epsilon_0|x|^3}$, and $\vec{\nabla}\cdot\vec{E} = -\nabla^2 V = q\delta^3(\vec{x})$,
so the function $\phi(\vec{x}) = 1/|x|$ has $\vec{\nabla}\phi = -\vec{x}/|x|^3$, $\nabla^2\phi = -4\pi\delta^3(\vec{x})$.
On the other hand, $W := e^{\pm ik|x|}$ satisfies $\vec{\nabla} W = \pm ik\dfrac{\vec{x}}{|x|}W$ and

$$\begin{aligned}
\nabla^2 W &= \left[\pm ik\left(\frac{\vec{\nabla}\cdot\vec{x}}{|x|} - \vec{x}\cdot\vec{\nabla}\frac{1}{|x|}\right)W - k^2\frac{\vec{x}^{\,2}}{|x|^2}\right]W \\
&= \left[\pm ik\left(\frac{3}{|x|} - \frac{\vec{x}^{\,2}}{|x|^3}\right) - k^2\right]W = \pm\frac{2ik}{|x|}W - k^2 W.
\end{aligned}$$

Thus

$$\begin{aligned}
\left(\nabla_x^2 + k^2\right)W\phi &= \left(\nabla_x^2 W\right)\phi + 2(\vec{\nabla}W)\cdot(\vec{\nabla}\phi) + W\left(\nabla_x^2\phi\right) + k^2 W\phi \\
&= \pm\frac{2ik}{|x|}W\phi \mp 2ik\frac{\vec{x}}{|x|}W\cdot\frac{\vec{x}}{|x|^3} - 4\pi W\delta^3(\vec{x}) \\
&= -4\pi\delta^3(\vec{x})
\end{aligned}$$

as $W(\vec{x})\delta^3(\vec{x}) = W(\vec{0})\delta^3(\vec{x}) = \delta^3(\vec{x})$. But the operator $(\nabla_x^2 + k^2)$ is translation-invariant, so we may translate the solution $W\phi/4\pi$ for $\vec{x}\,' = 0$ to arbitrary $\vec{x}\,'$,

$$G(\vec{x},\vec{x}\,') = \frac{e^{\pm ik|\vec{x}-\vec{x}\,'|}}{4\pi|\vec{x}-\vec{x}\,'|}. \tag{3}$$

We have ignored the issue of boundary conditions in this discussion. In general, satisfying the boundary conditions determines the added solutions of the homogeneous equation. For our purposes we will ask for outgoing waves, so we choose the upper sign, $e^{+ik|\vec{x}-\vec{x}\,'|}$. Thus the solution of (1) for Ψ is $\Psi(\vec{x}) = \int d^3x' G(\vec{x},\vec{x}\,')s(\vec{x}\,')$.

We want to reexpress the Green's function in spherical coordinates. If we solve the Green's function equation in spherical coordinates,

$$\begin{aligned}
\left(\nabla_x^2 + k^2\right)G(\vec{x},\vec{x}\,') &= -\delta(\vec{x}-\vec{x}\,') \\
&= -\frac{1}{r^2\sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi') \\
= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}L^2 + k^2\right)G(\vec{x},\vec{x}\,') &= -\frac{\delta(r-r')}{r^2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi),
\end{aligned}$$

where $r = |\vec{x}|$ and we have used the completeness relation (J3.56). If we let $G(\vec{x}, \vec{x}\,') = \sum_{\ell m} R_{\ell m}(r, \vec{x}\,') Y_{\ell m}(\theta, \phi)$ we have

$$\sum_{\ell m} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} + k^2 \right) R_{\ell m}(r, \vec{x}\,') Y_{\ell m}(\theta, \phi)$$
$$= -\frac{1}{r^2}\delta(r-r') \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi),$$

so we see that $R(r, \vec{x}\,') = \sum_\ell g_\ell(r, r') Y^*_{\ell m}(\theta', \phi')$ where

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} + k^2 \right) g_\ell(r, r') = -\frac{1}{r^2}\delta(r - r').$$

For $r \neq r'$ this is just the spherical Bessel equation, so the solutions are combinations of $j_\ell(kr)$ and $n_\ell(kr)$, or better of $j_\ell(kr)$ and $h^{(1)}_\ell := j_\ell(kr) + i n_\ell(kr) \to (-i)^{\ell+1} e^{ikr}/kr$. For $r < r'$ we need the solution to be regular at $r = 0$, so there are no n or h contributions, only j_ℓ,

$$g_\ell(r, r') = a_\ell(r') j_\ell(kr) \qquad \text{for } r < r',$$

while for $r > r'$ we want only outgoing waves, with e^{+ikr}, so the solution is pure $h^{(1)}_\ell$ with no $h^{(2)}_\ell$ (or j_ℓ)

$$g_\ell(r, r') = b_\ell(r') h^{(1)}_\ell(kr) \qquad \text{for } r > r'.$$

But from (3) we see that the Green's function is symmetric under $\vec{x} \leftrightarrow \vec{x}\,'$, so $a_\ell(r) = a_\ell h^{(1)}_\ell(kr)$ and $b_\ell(r) = a_\ell j_\ell(kr)$, and we may write more generally

$$g_\ell(r, r') = a_\ell j_\ell(kr_<) h^{(1)}_\ell(kr_>),$$

where $r_<$ is the smaller of r and r' and $r_>$ is the greater.

To determine the coefficients, observe that the derivative must be discontinuous, with

$$g'_\ell(r = r' + \epsilon) - g'_\ell(r = r' - \epsilon) \;=\; a_\ell k j_\ell(kr) h'^{(1)}_\ell(kr) - a_\ell k h^{(1)}_\ell(kr) j'_\ell(kr)$$
$$= \int_{r'-\epsilon}^{r'+\epsilon} \frac{-1}{r^2}\delta(r - r')\, dr = -\frac{1}{r'^2}$$

The first line is $k a_\ell$ times the Wronskian of $h^{(1)}_\ell$ and j_ℓ, which should be $-r^{-2}$. This agrees with the general statement that the Wronskian satisfies

$dW/dr = -P(r)W$, where $P(r)$ is the coefficient of the first order term, here $2/r$. Thus we can determine a_ℓ at any point, and as

$$\begin{aligned} j_\ell(x) &= \sqrt{\frac{\pi}{2x}}\, J_{\ell+1/2}(x) \to \frac{\sqrt{\pi}}{2\Gamma(\ell+3/2)}\left(\frac{x}{2}\right)^\ell, \\ n_\ell(x) &= \sqrt{\frac{\pi}{2x}}\, N_{\ell+1/2}(x) \to -\Gamma(\ell+1/2)\left(\frac{2}{x}\right)^{\ell+1}\frac{1}{2\sqrt{\pi}}, \\ h_\ell^{(1)} &= j_\ell + i n_\ell \to i n_\ell, \end{aligned}$$

so $j_\ell(r) h_\ell'^{(1)}(kr) - h_\ell^{(1)}(r) j_\ell'(kr) \to i/(kr)^2$, and $a_\ell = ik$. So all together

$$\frac{e^{ik|\vec{x}-\vec{x}\,'|}}{4\pi|\vec{x}-\vec{x}\,'|} = ik \sum_{\ell m} j_\ell(kr_<) h_\ell^{(1)}(kr_>) Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi).$$

So we are now ready to examine the solutions to the Helmholtz equation,

$$\begin{aligned} \vec{A}(\vec{x}) &= \frac{\mu_0}{4\pi}\int d^3x'\, \vec{J}(\vec{x}\,')\frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|} \\ &= i\mu_0 k \sum_{\ell m}\int d^3x'\, j_\ell(kr_<) h_\ell^{(1)}(kr_>) Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi)\vec{J}(\vec{x}\,'). \end{aligned}$$

If the sources are restricted to some region $|\vec{x}\,'| < d$, and we are asking about positions further from the origin, $r > d$, then $r_< = r'$ and $r_> = r$, and

$$\vec{A}(\vec{x}) = i\mu_0 k \sum_{\ell m} h_\ell^{(1)}(kr) Y_{\ell m}(\theta,\phi)\int d^3x'\, j_\ell(kr') Y^*_{\ell m}(\theta',\phi')\vec{J}(\vec{x}\,').$$

We see that $\vec{A}$ has an expansion in specified modes (ℓ, m) with the sources only determining the coefficients of these modes. If the source region is small compared to the wavelength, $d \ll \lambda = 2\pi/k = 2\pi c/\omega$, we have $kr' \ll 1$ whereever $\vec{J}(\vec{x}\,') \neq 0$, so we may use the expansion $j_\ell(x) \approx x^\ell/(2\ell+1)!!$, appropriate for $x \ll 1$. We see that the lowest ℓ value which contributes will dominate.

1.1 Zones

This expression can be simplified if we consider restrictions on the relative sizes of d, λ, and r.

If d and r are both much smaller than λ, we are in the *near* zone, we may set $k = 0$ while setting $kj_\ell(kr_<)h_\ell^{(1)}(kr>)$ to $\frac{-i}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}$. The fields are essentially instantaneously generated by the currents and charges. If, in addition we assume $d \ll r$, the lowest ℓ value will dominate.

If $r \gg \lambda$ and $r > d$, the fields oscillate rapidly with $h_\ell^{(1)}(kr) \to (-i)^{\ell+1}e^{ikr}/r$, falling off only as $1/r$, typical of radiation fields, and this is called the *far* or *radiation* zone. If we also have $d \ll \lambda$, $kr_<$ is small whereever $\vec{J}$ doesn't vanish, and the lowest ℓ mode will dominate.

We have not bothered to find $\Phi(\vec{x})$ because the Lorenz gauge $-i\omega\Phi/c^2 = -\vec{\nabla}\cdot\vec{A}$ gives it in terms of $\vec{A}$, except for $\omega = 0$, for which $\Phi(\vec{x})$ is given by the static Coulomb expression integrated over all the charges, the electric field is given by Coulombs law, and there is no magnetic field arising from Φ.

1.2 Electric Dipole

So, for example, if the $\ell = 0$ term does not vanish, we may write

$$\vec{A}(\vec{x}) \approx i\mu_0 k h_0^{(1)}(kr)Y_{00}\int d^3x' Y_{00}^*\vec{J}(\vec{x}\,') = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\vec{J}(\vec{x}\,'),$$

because $h_0^{(1)}(x) = -ie^{ix}/x$. As we are assuming all sources have an $e^{-i\omega t}$ time dependence, and the continuity equation tells us $\vec{\nabla}\cdot\vec{J} = -\partial\rho/\partial t = i\omega\rho$, we may write[1] $\int d^3x'\vec{J}(\vec{x}\,') = -i\omega\int d^3x'\,\vec{x}\,'\rho(\vec{x}\,')$. The integral is just the electric dipole moment, so

$$\vec{A}(\vec{x}) \approx -\frac{i\mu_0\omega}{4\pi}\,\vec{p}\,\frac{e^{ikr}}{r},$$

which is accurate for all $r > d$ to lowest order in d/λ, provided the dipole moment isn't zero.

Quite generally,

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A},$$

while *outside the region with sources*,

$$\frac{\partial\vec{D}}{\partial t} = -i\omega\epsilon_0\vec{E} = \vec{\nabla}\times\vec{H} \Longrightarrow \vec{E} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H},$$

[1] For any two vector functions $\vec{A}$ and $\vec{B}$, $\int_V A_i(\vec{\nabla}\cdot\vec{B}) = \sum_j\int_V(\partial_j(A_iB_j) - B_j\partial_jA_i) = \int_{\partial V}A_i(\vec{B}\cdot d\vec{S}) - \int_V\vec{B}\cdot\vec{\nabla}A_i$. So $\int_V\vec{A}(\vec{\nabla}\cdot\vec{B}) = \int_{\partial V}\vec{A}(\vec{B}\cdot d\vec{S}) - \int_V(\vec{B}\cdot\vec{\nabla})\vec{A}$. Let $\vec{A} = \vec{x}\,'$, $\vec{B} = \vec{J}$, and with $\vec{J}$ vanishing at infinity, we have $i\omega\int d^3x'\vec{x}\,'\rho(\vec{x}\,') = \int d^3x'\vec{x}\,'\vec{\nabla}\cdot\vec{J}(\vec{x}\,') = -\int d^3x'(\vec{J}\cdot\vec{\nabla})\vec{x}\,') = -\int d^3x'\vec{J}(\vec{x}\,')$.

with $Z_0 = \sqrt{\mu_0/\epsilon_0}$. The curl of $\vec{p}f(r)$ is $\hat{r} \times \vec{p}\,\partial f/\partial r$, so for our electric dipole source, we have

$$\vec{H} = \frac{ck^2}{4\pi}\,\hat{r} \times \vec{p}\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r},$$

and[2]

$$\begin{aligned}
\vec{E}(\vec{x}) &= i\frac{Z_0}{k}\frac{ck^2}{4\pi}\vec{\nabla} \times (\vec{x} \times \vec{p})\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r^2} \\
&= i\frac{k}{4\pi\epsilon_0}\left[-2p\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r^2} + \vec{x} \times (\vec{x} \times \vec{p})\frac{1}{r}\frac{d}{dr}\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r^2}\right] \\
&= i\frac{k}{4\pi\epsilon_0}\frac{e^{ikr}}{r^2}\frac{1}{ikr}\left[2\vec{p}(1 - ikr) + \vec{x} \times (\vec{x} \times \vec{p})\left(\frac{3}{r^2} - \frac{3ik}{r} - k^2\right)\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left\{-k^2\hat{r} \times (\hat{r} \times \vec{p}) + [3\hat{r}(\hat{r}\cdot\vec{p}) - \vec{p}]\left(\frac{1}{r^2} - \frac{ik}{r}\right)\right\}.
\end{aligned}$$

Note the first term in $\vec{E}$ is perpendicular to $\vec{x}$, but the second is not. However this longitudinal term falls off as r^{-2}, so may be neglected in the radiation zone $r \gg \lambda$, where we can write

$$\left.\begin{aligned}
\vec{H} &= \frac{ck^2}{4\pi}\,\hat{r} \times \vec{p}\,\frac{e^{ikr}}{r} \\
\vec{E} &= \frac{-k^2 e^{ikr}}{4\pi\epsilon_0 r}\hat{r} \times (\hat{r} \times \vec{p}) = -Z_0\hat{r} \times \vec{H}
\end{aligned}\right\} \qquad \text{in the } \textbf{radiation zone}.$$

In the *near zone*, that is when $d < r \ll \lambda$, we have

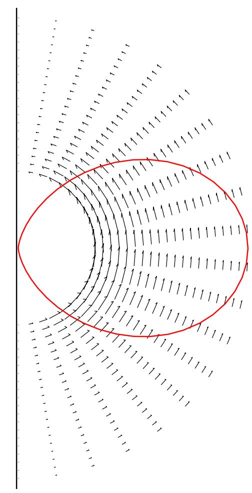
$$\left.\begin{aligned}
\vec{H} &= \frac{i\omega}{4\pi r^2}\hat{r} \times \vec{p} \\
\vec{E} &= \frac{1}{4\pi\epsilon_0 r^3}\left(3\hat{r}(\hat{r}\cdot\vec{p}) - \vec{p}\right)
\end{aligned}\right\} \qquad \text{in the } \textbf{near zone}$$

The electric field in the near zone is just what we would have from a static dipole of the present value at each moment, and the E field dominates the H field in this zone.

[2] $\vec{\nabla} \times (\vec{x} \times \vec{p}f(r))_i = \sum_{jkmq} \epsilon_{ijk}\partial_j\epsilon_{kmq}x_m p_q f(r) = \sum_j \partial_j x_i p_j f(r) - \partial_j x_j p_i f(r) = -2p_i f(r) + \sum_j (x_i p_j - x_j p_i)\hat{r}_j df/dr(r)$, so $\vec{\nabla} \times (\vec{x} \times \vec{p}f(r)) = -2\vec{p}f(r) + \vec{x} \times (\vec{x} \times \vec{p})r^{-1}df/dr$.

In the intermediate zone, where r is comparable to λ and kr is of order 1, all of the terms in Jackson Eq. 9.18 are comparable, and the fields have no particularly simple approximate expression.

If we ask about the power radiated at large distances, the average power per unit solid angle is



$$\begin{aligned}\frac{\langle P\rangle}{d\Omega} &= \frac{r^2}{2}\,\mathrm{Re}\,\hat{r}\cdot(\vec{E}\times\vec{H}^*) \approx \frac{Z_0c^2k^4}{2(4\pi)^2}\left|\hat{r}\times(\hat{r}\times\vec{p})\right|^2 \\ &= \frac{Z_0c^2k^4}{32\pi^2}\,p^2\,(1-\cos^2\theta) = \frac{Z_0c^2k^4}{32\pi^2}\,p^2\,\sin^2\theta,\end{aligned}$$

where θ is the angle between $\vec{p}$ and $\vec{x}$. The total power radiated is

$$\begin{aligned}\langle P\rangle &= 2\pi\int_0^\pi d\theta\,\sin\theta\frac{\langle P\rangle}{d\Omega} \\ &= \frac{Z_0c^2k^4}{16\pi}\,p^2\int_0^\pi d\theta\,\sin^3\theta = \frac{Z_0c^2k^4}{12\pi}\,p^2.\end{aligned}$$

1.3 The Next Order

To include the next order contributions, essential if the dipole moment vanishes, we look at the $\ell=1$ term in the expansion,

$$\vec{A}^{(1)} = i\mu_0 k h_1^{(1)}(kr)\sum_{m=-1}^{m=1}Y_{1m}(\theta,\phi)\int d^3x'\,j_1(kr')Y^*_{1m}(\theta',\phi')\vec{J}(\vec{x}\,').$$

With

$$h_1^{(1)}(x) = -\frac{e^{ix}}{x}\left(1+\frac{i}{x}\right),\qquad j_1(x)=\frac{x}{3}\left(1+\mathcal{O}x^2\right),$$

and

$$\sum_{m=-1}^{m=1}Y_{1m}(\theta,\phi)Y^*_{1m}(\theta',\phi') = \frac{3}{4\pi}\hat{r}\cdot\hat{r'},$$

we see that

$$\vec{A}^{(1)} = i\mu_0 k\frac{3}{4\pi}\frac{1}{3}\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\int d^3x'\,\hat{r}\cdot\vec{x}\,'\vec{J}(\vec{x}\,').$$

The multipole moments involved here are tensors $\sim\vec{x}\,'\vec{J}(\vec{x}\,')$. The antisymmetric part is the integral of the magnetization

$$\mathcal{M}(\vec{x}\,') = \frac{1}{2}\vec{x}\,'\times\vec{J}(\vec{x}\,'),\quad \text{with } \vec{m}=\int d^3x'\,\mathcal{M}(\vec{x}\,')$$

the magnetic dipole moment.

$$\hat{r} \times \vec{m} = \frac{1}{2} \int d^3x' \left[(\hat{r} \cdot \vec{J}(\vec{x}\,')\vec{x}\,' - (\hat{r} \cdot \vec{x}\,')\vec{J}(\vec{x}\,') \right].$$

The symmetric piece is related to the electric quadripole moment

$$\begin{aligned} Q_{ij} &:= \int d^3x' \left(3x'_i x'_j - x'^2 \delta_{ij} \right) \rho(\vec{x}\,') \\ &= \int d^3x' \left(3x'_i x'_j - x'^2 \delta_{ij} \right) \frac{-i}{\omega} \vec{\nabla} \cdot \vec{J} \\ &= \frac{i}{\omega} \int d^3x' J_k(\vec{x}\,') \partial'_k \left(3x'_i x'_j - x'^2 \delta_{ij} \right) \\ &= \frac{i}{\omega} \int d^3x' J_k(\vec{x}\,') \left(3\delta_{ik} x'_j + 3\delta_{jk} x'_i - 2x'_k \delta_{ij} \right) \end{aligned}$$

So $\hat{r} \cdot Q = \frac{i}{\omega} \int d^3x' \left(3\hat{r} \cdot \vec{J}(\vec{x}\,')\,\vec{x}\,' + 3\hat{r} \cdot \vec{x}\,'\,\vec{J}(\vec{x}\,') - 2\vec{x}\,' \cdot \vec{J}(\vec{x}\,')\,\hat{r} \right).$

For completeness we need to consider a electric monopole term

$$M_E = \int d^3x' x'^2 \rho(\vec{x}\,') = \frac{2i}{\omega} \int d^3x' \vec{x}\,' \cdot \vec{J}.$$

So our complete $\ell = 1$ vector potential is

$$\vec{A}^{(1)} = -i \frac{\mu_0 k}{24\pi} \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr} \right) (6\hat{r} \times \vec{m} + i\omega \hat{r} \cdot \mathbf{Q} + i\omega M_E \hat{r}).$$

Let us evaluate $\vec{H}$ and $\vec{E}$ only to leading order in $1/r$, so we need only consider the derivative acting on e^{ikr}, and needn't worry about $\vec{\nabla} \times \hat{r}$. We can also drop the i/kr term.

Then

$$\vec{H}^{(1)} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}^{(1)} = \frac{k^2}{24\pi} \frac{e^{ikr}}{r} \hat{r} \times (6\hat{r} \times \vec{m} + i\omega \hat{r} \cdot \mathbf{Q} + i\omega M_E \hat{r}).$$

The electric monopole contribution vanishes due to $\hat{r} \times \hat{r} = 0$. The magnetic dipole contributes

$$\begin{aligned} \vec{H}^{\text{MD}} &= \frac{k^2}{4\pi} \frac{e^{ikr}}{r} \hat{r} \times (\hat{r} \times \vec{m}), \\ \vec{E}^{\text{MD}} &= \frac{iZ_0}{k} \vec{\nabla} \times \vec{H}^{\text{MD}} = -\frac{k^2 Z_0}{4\pi} \frac{e^{ikr}}{r} \hat{r} \times (\hat{r} \times (\hat{r} \times \vec{m})) \\ &= \frac{k^2 Z_0}{4\pi} \frac{e^{ikr}}{r}\, \hat{r} \times \vec{m}. \end{aligned}$$

These are of the same form as for the electric dipole, but with $\vec{E}$ and $\vec{H}$ interchanged. The radiation pattern is the same, but the polarization has $\vec{E} \perp \vec{m}$ here, while $\vec{E}$ lies in the plane including $\hat{r}$ and $\vec{p}$ for the electric dipole.

One might be tempted to think the electric quadripole also vanishes, as it involves $\hat{r} \times (\hat{r} \cdot \mathbf{Q})$, and $\mathbf{Q}$ is symmetric. But that is incorrect: in $\hat{r} \times (\hat{r} \cdot \mathbf{Q}) = \sum \epsilon_{ijk} \hat{r}_i Q_{j\ell} \hat{r}_\ell \hat{e}_k$, the summands are symmetric under $i \leftrightarrow \ell$ and antisymmetric under $i \leftrightarrow j$, but that does not make things vanish. Jackson defines the vector $\vec{Q}(\vec{n}) := \sum Q_{ij} n_j \hat{e}_i$, and then we have $\hat{r} \times \vec{Q}(\hat{r})$. Then again keeping only $1/r$ terms,

$$\begin{aligned} \vec{H}^{\rm EQ} &= \frac{ick^3}{24\pi} \frac{e^{ikr}}{r} \hat{r} \times \vec{Q}(\hat{r}) \\ \vec{E}^{\rm EQ} &= \frac{-iZ_0 ck^3}{24\pi} \frac{e^{ikr}}{r} \hat{r} \times \left(\hat{r} \times \vec{Q}(\hat{r}) \right). \end{aligned}$$

Power radiated

Probably the most interesting thing one might ask is how much power is radiated, and in which directions, as we did for the electric dipole.

For an electric quadripole, $|\mathbf{Q}|$ is a symmetric real traceless tensor, so we could rotate the coordinate system so that it will be diagonal. If we take an axially symmetric case, with $Q_{zz} = -2Q_{xx} > 0$.
The average power per unit solid angle is

$$\begin{aligned} \frac{\langle P \rangle}{d\Omega} &= \frac{r^2}{2} \,\mathrm{Re}\, \hat{r} \cdot (\vec{E} \times \vec{H}^*) \\ &\propto \left| \hat{r} \times (\hat{r} \times \mathbf{Q}(\hat{r})) \right|^2 \end{aligned}$$

as shown.