Physics 504, Lecture 9
Feb. 21, 2011

1 Ionosphere, Redux

Let us return to the resonant cavity formed by the surface of the Earth a spherical shell of radius $r = R_E$, and the ionosphere, treated as a spherical shell of radius $r = R_E + h$. We will treat both as perfect conductors, though by lab-scale standards they are far from that, so the fields will obey $H_r = 0$ and $E_\theta = E_\phi$ at $r = R_E$ and at $R = R_E + h$. Between the shells the fields obey Maxwell's equations, and because our equations are linear and time-independent, we can work with one fourier mode at a time, assuming all fields are $\propto e^{-i\omega t}$. We will treat the region between the shells as vacuum, as the permittivity and permeability of air differs little from ϵ_0 and μ_0. Thus

$$\vec\nabla \times \vec E = ikZ_0\vec H, \quad \vec\nabla \times \vec H = -i\frac{k}{Z_0}\vec E, \quad \vec\nabla\cdot\vec E = 0, \quad \vec\nabla\cdot\vec H = 0,$$

where $k = \omega/c$ is not exactly a component of a wavenumber, but what it would be for a plane wave, and $Z_0 = \sqrt{\mu_0/\epsilon_0}$ and $c = 1/\sqrt{\mu_0\epsilon_0}$ are the impediance of free space and the speed of light in vacuum. Then

$$\vec\nabla \times \left(\vec\nabla \times \vec H\right) = \vec\nabla \times \left(-i\frac{k}{Z_0}\vec E\right) = k^2\vec H,$$

but as $\vec\nabla \times \left(\vec\nabla \times \vec V\right) = \vec\nabla\left(\vec\nabla\cdot\vec V\right) - \nabla^2\vec V$ for any vector, and as $\vec\nabla\cdot\vec H = 0$, we have

$$\left(\nabla^2 + k^2\right)\vec H = 0, \quad \vec\nabla\cdot\vec H = 0, \quad \text{and } \vec E = i\frac{Z_0}{k}\vec\nabla \times \vec H.$$

Similiarly we can derive

$$\left(\nabla^2 + k^2\right)\vec E = 0, \quad \vec\nabla\cdot\vec E = 0, \quad \text{and } \vec H = -i\frac{1}{kZ_0}\vec\nabla \times \vec E.$$

In particular each cartesian component of $\vec E$ and $\vec H$ obeys the Helmholtz equation, but this is not convenient for a spherically symmetric situation. But just as for waveguides we could consider the z component of one of $\vec E$ or $\vec H$ to determine the rest, the analogue here is the radial component, or rather $\vec r\cdot\vec A$, where $\vec A$ is either $\vec E$ or $\vec H$. In general,

$$\nabla^2(\vec r\cdot\vec A) = \sum_{ij}\frac{\partial^2}{\partial r_i^2}(r_jA_j) = \sum_{ij}\left(r_j\frac{\partial^2}{\partial r_i^2}A_j + 2\frac{\partial A_j}{\partial r_i}\delta_{ij}\right) = \vec r\cdot\nabla^2\vec A + 2\vec\nabla\cdot\vec A.$$

But for either $\vec E$ or $\vec H$, the divergence vanishes, so

$$\left(\nabla^2 + k^2\right)(\vec r\cdot\vec E) = 0, \qquad \left(\nabla^2 + k^2\right)(\vec r\cdot\vec H) = 0.$$

We will define two modes, a magnetic multipole field for which $\vec r\cdot\vec E = 0$, and a electric multipole field for which $\vec r\cdot\vec H = 0$. Whichever one is not identically zero must satisfy the Helmholtz equation, which can be solved by separation of variables. Either

$$\begin{aligned} \vec r\cdot\vec H^{(M)}_{\ell m} &= \frac{\ell(\ell+1)}{k}g_\ell(kr)Y_{\ell m}(\theta,\phi), & \vec r\cdot\vec E^{(M)} &= 0 \\ \text{or}\quad \vec r\cdot\vec E^{(E)}_{\ell m} &= -Z_0\frac{\ell(\ell+1)}{k}f_\ell(kr)Y_{\ell m}(\theta,\phi), & \vec r\cdot\vec H^{(E)} &= 0. \end{aligned}$$

We recognize the spherical harmonics from Quantum Mechanics, the solutions of the differential equation

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\theta^2}\right]Y_{\ell m} = \ell(\ell+1)Y_{\ell m}.$$

Actually we can be more intuitive, for we know from quantum mechanics that if we define the operators

$$\vec L = -i\vec r\times\vec\nabla, \quad L_\pm = L_x \pm iL_y = e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right), \quad L_z = -i\frac{\partial}{\partial\phi},$$

that in terms of these, we have

$$L_\pm Y_{\ell m} = \sqrt{(\ell\mp m)(\ell\pm m+1)}Y_{\ell,m\pm1}, \quad L_zY_{\ell m} = mY_{\ell m}, \quad L^2Y_{\ell m} = \ell(\ell+1)Y_{\ell m}.$$

Dotting $\vec r$ into the curl equations we have

$$ikZ_0\vec r\cdot\vec H = \vec r\cdot\left(\vec\nabla\times\vec E\right) = \left(\vec r\times\vec\nabla\right)\cdot\vec E = i\vec L\cdot\vec E.$$

Thus for the magnetic multipole field

$$\vec L\cdot\vec E^{(M)}_{\ell m} = kZ_0\vec r\cdot\vec H = Z_0g_\ell(kr)L^2Y_{\ell m},$$

which at least hints at

$$\vec{E}^{(M)}_{\ell m} = Z_0 g_\ell(kr)\vec{L}Y_{\ell m},$$

Also, this is consistent with $\vec{r}\cdot\vec{E}^{(M)}_{\ell m} = 0$ as $\vec{r}\cdot\vec{L} = -i\vec{r}\cdot\left(\vec{r}\times\vec{\nabla}\right) = 0$. The rest of the fields in a magnetic multipole are

$$\vec{H}^{(M)}_{\ell m} = -\frac{i}{kZ_0}\vec{\nabla}\times\vec{E}^{(M)}_{\ell m}.$$

This magnetic multipole field configuration is also called transverse electric (TE), as $\vec{E}$ is transverse to the radial direction.

Of course a similar calculation determines the fields of an electric multipole, or TM mode. Then

$$\begin{aligned} \vec{H}^{(E)}_{\ell m} &= f_\ell(kr)\vec{L}Y_{\ell m}(\theta,\phi), && (1)\\ \vec{E}^{(E)}_{\ell m} &= i\frac{Z_0}{k}\vec{\nabla}\times\vec{H}^{(E)}_{\ell m} = \frac{Z_0}{k}\vec{\nabla}\times\left(\vec{r}\times\vec{\nabla}\right)f_\ell(kr)Y_{\ell m}(\theta,\phi). && (2)\end{aligned}$$

Now[1] $\vec{\nabla}\times\left(\vec{r}\times\vec{\nabla}\right) = \vec{r}\nabla^2 - \vec{\nabla}\left(1+r\dfrac{\partial}{\partial r}\right)$.

If we now want to extract the transverse piece of the electric field, we can note that

$$\begin{aligned} \vec{r}\times E^{(E)}_{\ell m} &= \frac{Z_0}{k}\vec{r}\times\left(\vec{r}\nabla^2 - \vec{\nabla}\left(1+r\frac{\partial}{\partial r}\right)\right)f_\ell(kr)Y_{\ell m}(\theta,\phi)\\ &= -\frac{Z_0}{k}\left(\vec{r}\times\vec{\nabla}\right)\left(1+r\frac{\partial}{\partial r}\right)f_\ell(kr)Y_{\ell m}(\theta,\phi)\\ &= -i\frac{Z_0}{k}\left[\left(1+r\frac{\partial}{\partial r}\right)f_\ell(kr)\right]\left[\vec{L}\,Y_{\ell m}(\theta,\phi)\right].\end{aligned}$$

First note that for $\ell = 0$, the entire magnetic field $\vec{H}^{(E)}$ or the entire electric field $\vec{E}^{(M)}$ vanishes everywhere, but then, for $k\neq 0$, so does the other field, so we have no mode of oscillation. For $\ell\neq 0$ the angular part

[1] $\left(\vec{\nabla}\times\left(\vec{r}\times\vec{\nabla}\right)\right)_i = \epsilon_{ijk}\partial_j\epsilon_{k\ell m}r_\ell\partial_m = \left(\delta_{i\ell}\delta_{jm}-\delta_{im}\delta_{j\ell}\right)\left(\delta_{j\ell}\partial_m + r_\ell\partial_j\partial_m\right)$
$= \partial_i - 3\partial_i + r_i\nabla^2 - \left(\vec{r}\cdot\vec{\nabla}\right)\partial_i = r_i\nabla^2 - 2\partial_i - \partial_i\left(\vec{r}\cdot\vec{\nabla}\right) + \partial_i$
$= r_i\nabla^2 - \partial_i\left(1+r\dfrac{\partial}{\partial r}\right)$.

$\vec{L}Y_{\ell m}(\theta,\phi)$ will not vanish, so vanishing of the transverse parts of $E^{(E)}_{\ell m}$ requires $\left(1+r\frac{\partial}{\partial r}\right)f_\ell(kr) = 0$

So the conditions for our resonant cavity requires either $g(kr)$ or $f(kr) + rdf/dr$ to vanish at both $r = R_E$ and $r = R_E + h$.

The radial part of the functions comes from the separation of variables. As

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}L^2,$$

the radial dependence of an (ℓ, m) mode, which is an eigenfunction of L^2 with eigenvalue $\ell(\ell+1)$, satisfies

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} + k^2\right)g_\ell(kr) = 0.$$

The same equation holds for $f_\ell(kr)$. It turns out the solutions are spherical Bessel and Hankel functions, which behave somewhat like $\sin(kr)$ or $\cos(kr)$ dampled by $1/r$. Thus it is easy to make combinations of these which will vanish at the two points, if we have k of the order of π/h. This corresponds to a frequency of about 10 kHz, well below radio frequencies. Thus radio frequency waves can be considered in the geometrical optics approximation, (*i.e.* ray tracing) appropriate when the wavelength is short compared to the dimensions of the geometry.

But we are also interested in the possibility of much longer wavelengths, comparable to the Earth's radius. The book's argument is not very clear, so let us examine the radial equation

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2}\right)f_\ell(r) = 0,$$

which is also the equation satisfied by $g_\ell(r)$. There are several useful ways of transforming the equation, by setting $x = \beta kr$ and

$$\begin{aligned} f_\ell(r) &= \frac{u_{\ell,\alpha,\beta}(\beta kr)}{(\beta kr)^\alpha}\\ &\Longrightarrow \left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + \frac{1}{\beta^2} - \frac{\ell(\ell+1)}{x^2}\right)\frac{u_{\ell,\alpha,\beta}(x)}{x^\alpha} = 0\\ &\Longrightarrow \left(\frac{d^2}{dx^2} + \frac{2(1-\alpha)}{x}\frac{d}{dx} + \frac{1}{\beta^2} + \frac{\alpha(\alpha-1)-\ell(\ell+1)}{x^2}\right)u_{\ell,\alpha,\beta}(x) = 0.\end{aligned}$$

We will consider two useful choices of α and β. First, let $\alpha = 1/2, \beta = 1$, giving

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{(\ell+1/2)^2}{x^2}\right) u_{\ell,\frac{1}{2},1}(x) = 0,$$

which we recognize as Bessel's equation with $\nu = \ell + \frac{1}{2}$, with solutions $J_{\ell+1/2}(x)$ and $N_{\ell+1/2}(x)$, the Bessel and Neumann functions. These functions are well understood, but we also can define the spherical Bessel and spherical Neumann functions

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell+1/2}(x), \qquad n_\ell(x) = \sqrt{\frac{\pi}{2x}} N_{\ell+1/2}(x)$$

$$h_\ell^{(1,2)}(x) = \sqrt{\frac{\pi}{2x}} \left(J_{\ell+1/2}(x) \pm i\, N_{\ell+1/2}(x)\right).$$

Another interesting choice is $\alpha = 1$, $\beta = 1/\sqrt{\ell(\ell+1)}$, so the equation becomes

$$\left(\frac{d^2}{dx^2} + \ell(\ell+1)\left(1 - \frac{1}{x^2}\right)\right) u_\ell = 0,$$

and the boundary conditions at the ends of the interval, $\beta k R_E$ and $\beta k(R_E+h)$ are $(1 + r d/dr)(u(\beta kr)/\beta kr) = 0 = du/dx$ with $x = \beta kr$. We see that if the range of x is such that 1 lies roughly in the middle, the second derivative will change sign, and it is possible to fix things such that $du/dx = 0$ at the ends of the interval with very little variation in $u(x)$. To lowest order this would ask to have $\beta k(R_E + h/2) = 1$, or

$$k = \frac{\sqrt{\ell(\ell+1)}}{R_E + h/2}, \qquad f = \frac{\omega}{2\pi} = \frac{c}{2\pi}\frac{\sqrt{\ell(\ell+1)}}{R_E + h/2} = 7.42\sqrt{\ell(\ell+1)} \text{ Hz}.$$

Note the same possibility does *not* exist for the magnetic multipole (TE) mode, as for g to vanish twice a distance h apart, along with dg/dx somewhere between them, requires h to be, roughly speaking, half a wavelength.