Physics 504, Lecture 7
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1 Energy Flow, Density and Attenuation

We have seen that there are discrete modes λ for electromagnetic waves with $\vec{E},\vec{B} \propto e^{ikz-i\omega t}$ corresponding, for real ω and k, to waves travelling in the z direction, with a dispersion relation $k^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2$, with discrete values of γ_λ for TE or TM modes, or $\gamma = 0$ for TEM modes. This relation for TE and TM modes are of the same form as for the high-frequency behavior of dielectrics, (J 7.61) with $\gamma/\sqrt{\mu\epsilon}$ playing the roll of the plasma frequency. The phase velocity is

$$v_p = \omega/k = \frac{1}{\sqrt{\mu\epsilon}}\frac{1}{\sqrt{1-\left(\frac{\omega_\lambda}{\omega}\right)^2}} > \frac{1}{\sqrt{\mu\epsilon}},$$

greater than the velocity in an infinite medium ($\mathbb{R}^3$), as for all travelling waves we have $\omega > \omega_\lambda$.

On the other hand, as $k^2 - \mu\epsilon\omega^2 =$ constant, we have $kdk = \mu\epsilon\omega d\omega$, so the group velocity

$$v_g = \frac{d\omega}{dk} = \frac{1}{\mu\epsilon}\frac{k}{\omega} = \frac{1}{\mu\epsilon}\frac{1}{v_p} < \frac{1}{\sqrt{\mu\epsilon}},$$

less than the infinite medium velocity.

1.1 Energy flow and density

These calculations assumed the walls had infinite conductivity, but generally the walls are good but not perfect conductors,. Then there will be eddy currents and energy loss in the walls, and a right-moving wave will be attenuated in the z direction, so k will develop a small positive imaginary part. We may estimate this attenuation constant by comparing the energy travelling past z to the energy per unit length lost at z. The energy flux is given by the z component of the Poynting vector, so the power transmitted down the pipe is $P = \int_A \hat{z}\cdot\vec{S}$.

Up to now we have worked with quantities linear in $\vec{E}$ or $\vec{H}$ so the proviso that the real fields are just the real parts of our complex fields could be swept under the rug. But the Poynting vector $\vec{S} = \vec{E}_{\rm phys}\times\vec{H}_{\rm phys}$ is quadratic. Our wave propagating in the z direction is actually

$$\vec{E}_{\rm phys}(x,y,z,t) = \frac{1}{2}\left(\vec{E}(x,y,k,\omega)e^{ikz-i\omega t} + \vec{E}^*(x,y,k,\omega)e^{-ikz+i\omega t}\right).$$

So when we talk in terms of a wave with a single k and ω,

$$\vec{E}_{\rm phys}(x,y,z,t) = \text{ Re } \vec{E}(x,y,z,t) = \text{ Re }\left(\vec{E}(x,y,k,\omega)e^{ikz-i\omega t}\right),$$

we have

$$\begin{aligned}
\vec{S}_{\rm phys} &= \vec{E}_{\rm phys}\times\vec{H}_{\rm phys}\\
&= \frac{1}{4}\Bigg(\left(\vec{E}(x,y,k,\omega)e^{ikz-i\omega t} + \vec{E}^*(x,y,k,\omega)e^{-ikz+i\omega t}\right)\times\\
&\qquad\qquad\left(\vec{H}(x,y,k,\omega)e^{ikz-i\omega t} + \vec{H}^*(x,y,k,\omega)e^{-ikz+i\omega t}\right)\Bigg)\\
&= \frac{1}{4}\Bigg(\vec{E}(x,y,k,\omega)\times\vec{H}(x,y,k,\omega)e^{2ikz-2i\omega t}\\
&\qquad +\vec{E}^*(x,y,k,\omega)\times\vec{H}(x,y,k,\omega) + \vec{E}(x,y,k,\omega)\times\vec{H}^*(x,y,k,\omega)\\
&\qquad +\vec{E}^*(x,y,k,\omega)\times\vec{H}^*(x,y,k,\omega)e^{-2ikz+2i\omega t}\Bigg)
\end{aligned}$$

The first and last term are rapidly oscillating, so if we are interested in the average of $\vec{S}$, we have

$$\begin{aligned}
\langle\vec{S}\rangle &= \frac{1}{4}\left(\vec{E}^*(x,y,k,\omega)\times\vec{H}(x,y,k,\omega) + \vec{E}(x,y,k,\omega)\times\vec{H}^*(x,y,k,\omega)\right)\\
&= \frac{1}{2}\text{ Re }\left(\vec{E}(x,y,k,\omega)\times\vec{H}^*(x,y,k,\omega)\right)
\end{aligned}$$

The power transmitted down the waveguide is the integral of the z component of this, so only the transverse components of $\vec{E}$ and $\vec{H}$ are needed, and these are given by

$$\begin{array}{lllll}
\text{TM:} & E_z = \psi, & \vec{E}_t = i\dfrac{k}{\gamma_\lambda^2}\vec{\nabla}_t\psi, & \vec{H}_t = i\dfrac{\epsilon\omega}{\gamma_\lambda^2}\hat{z}\times\vec{\nabla}_t\psi, & \psi|_S = 0\\
\text{TE:} & H_z = \psi, & \vec{H}_t = i\dfrac{k}{\gamma_\lambda^2}\vec{\nabla}_t\psi, & \vec{E}_t = -i\dfrac{\mu\omega}{\gamma_\lambda^2}\hat{z}\times\vec{\nabla}_t\psi, & \dfrac{\partial\psi}{\partial n}|_S = 0
\end{array}$$

with $(\nabla_t^2 + \gamma_\lambda^2)\psi = 0$. Thus the average transmitted power is

$$P = \hat{z} \cdot \ \mathrm{Re} \int_A S = \frac{\omega k}{2\gamma_\lambda^4} \int_A |\vec{\nabla}_t \psi|^2 \cdot \begin{cases} \epsilon & \text{(for TM)} \\ \mu & \text{(for TE)} \end{cases}$$

The integral

$$\int_A |\vec{\nabla}_t \psi|^2 = \oint_S \psi^* \frac{\partial \psi}{\partial n} - \int_A \psi^* \nabla_t^2 \psi = 0 + \gamma_\lambda^2 \int_A \psi^* \psi.$$

With $\omega_\lambda := \gamma_\lambda / \sqrt{\mu\epsilon}$, $k = \omega\sqrt{\mu\epsilon}\sqrt{1 - \omega_\lambda^2/\omega^2}$, this is

$$P = \frac{1}{2\sqrt{\mu\epsilon}} \left(\frac{\omega}{\omega_\lambda}\right)^2 \sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} \int_A \psi^* \psi \cdot \begin{cases} \epsilon & \text{(for TM)} \\ \mu & \text{(for TE)} \end{cases}$$

We might also calculate the energy per unit length in the waveguide, $U = \int_A u = \frac{1}{2}\int_A \left(\vec{E}_{\text{phys}} \cdot \vec{D}_{\text{phys}} + \vec{B}_{\text{phys}} \cdot \vec{H}_{\text{phys}}\right)$, so

$$\langle U \rangle = \frac{1}{4} \int_A \epsilon |\vec{E}|^2 + \mu |\vec{H}|^2$$

Here the longitudinal ($\hat{z}$) components enter as well as the transverse ones. For the TM mode, we have

$$\begin{aligned} |\vec{H}|^2 = |\vec{H}_t|^2 &= \left(\frac{\epsilon\omega}{\gamma_\lambda^2}\right)^2 \left|\vec{\nabla}_t \psi\right|^2 \\ |\vec{E}|^2 = |\vec{E}_t|^2 + |\vec{E}_z|^2 &= \left(\frac{k}{\gamma_\lambda^2}\right)^2 \left|\vec{\nabla}_t \psi\right|^2 + |E_z|^2 \end{aligned}$$

As $E_z = \psi$,

$$\begin{aligned} \langle U \rangle &= \frac{1}{4}\left[\epsilon\left(\frac{k^2}{\gamma_\lambda^4}\int_A |\vec{\nabla}_t\psi|^2 + \int_A |\psi|^2\right) + \mu\left(\frac{\epsilon\omega}{\gamma_\lambda^2}\right)^2 \int_A |\vec{\nabla}_t \psi|^2\right] \\ &= \frac{\epsilon}{2}\frac{\mu\epsilon\omega^2}{\gamma_\lambda^2}\int_A |\psi|^2 = \frac{\epsilon}{2}\frac{\omega^2}{\omega_\lambda^2}\int_A |\psi|^2 \qquad \text{(TM mode).} \end{aligned}$$

Similarly for the TE mode, we need $\vec{H}_t = ik\gamma_\lambda^{-2}\vec{\nabla}_t\psi$ and $H_z = \psi$

$$\int_A |\vec{H}|^2 = \frac{k^2}{\gamma_\lambda^4}\int_A \left|\vec{\nabla}\psi\right|^2 + \int_A |\psi|^2 = \left(\frac{k^2}{\gamma_\lambda^2} + 1\right)\int_A |\psi|^2 = \frac{\omega^2}{\omega_\lambda^2}\int_A |\psi|^2$$

while

$$\int_A |\vec{E}|^2 = \frac{\mu^2\omega^2}{\gamma_\lambda^4}\int_A |\vec{\nabla}\psi|^2 = \frac{\mu}{\epsilon}\frac{\omega^2}{\omega_\lambda^2}\int_A |\psi|^2,$$

so

$$\langle U \rangle = \frac{\mu}{2}\frac{\omega^2}{\omega_\lambda^2}\int_A |\psi|^2 \qquad \text{(TE mode).}$$

Note that in either case,

$$\frac{\langle P \rangle}{\langle U \rangle} = \frac{1}{\sqrt{\epsilon\mu}}\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} = v_g.$$

As we might expect, the energy flux is the energy density times the group velocity.

1.2 Attenuation

We now turn to the major effect of the finiteness of the conductivity σ in the walls of the waveguide. We saw in section 8.1 that the power loss per unit area of interface is given by

$$\frac{dP_{\text{loss}}}{dA} = \frac{1}{2\delta\sigma}\left|\vec{H}_\parallel\right|^2,$$

where $\delta = \sqrt{2/\mu_c\sigma\omega}$ is the skin depth. This gives a loss of power per unit length proportional to the square of the fields and therefore to the power, so the power transmitted along the waveguide will fall off exponentially. This can be described by giving k a small imaginary part, $\mathrm{Im}\, k = \beta_\lambda$, giving a factor $e^{-\beta_\lambda z}$ to each of the fields, and $e^{-2\beta_\lambda z}$ to the power flow. Thus

$$\frac{dP}{dz} = -2\beta_\lambda P(z) = -\frac{1}{2\sigma\delta}\oint_\Gamma \left|\hat{n} \times \vec{H}\right|^2 d\ell.$$

For the TM mode,

$$\hat{n} \times \vec{H} = \hat{n} \times \vec{H}_t = \frac{i\epsilon\omega}{\gamma_\lambda^2}\hat{n} \times (\hat{z} \times \vec{\nabla}_t\psi) = \frac{i\epsilon\omega}{\gamma_\lambda^2}\left(\hat{n} \cdot \vec{\nabla}_t \psi\right)\hat{z}$$

so

$$\beta_\lambda = \frac{1}{4\sigma\delta}\left(\frac{\epsilon\omega}{\gamma_\lambda^2}\right)^2 \int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \bigg/ \frac{\omega k \epsilon}{2\gamma_\lambda^4}\int_A \left|\vec{\nabla}\psi\right|^2 = \frac{\omega\epsilon}{2k\sigma\delta}\int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \bigg/ \int_A \left|\vec{\nabla}\psi\right|^2$$

Though we don't know the ratio of the integrals, they differ only in that numerator takes only the square of one of the two components of $\vec{\nabla}\psi$, and integrates it over the circumference rather than the area. Furthermore the ratio of these integrals depends only on the mode λ and not on the frequency ω of the transmitted wave. Let us write

$$\int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \Big/ \int_A \left|\vec{\nabla}\psi\right|^2 = \frac{C}{A}\xi_\lambda,$$

where C and A are the circumference and the area of the wave guide's cross section, and ξ_λ is a dimensionless number expected to be of order-of-magnitude 1. Let us display the frequency dependance explicitly by writing $\delta = \delta_\lambda\sqrt{\omega_\lambda/\omega}$, $k = \omega\sqrt{\mu\epsilon}\sqrt{1-\omega_\lambda^2/\omega^2}$. Thus

$$\beta_\lambda = \sqrt{\frac{\epsilon}{\mu}}\frac{1}{\sigma\delta_\lambda}\frac{C}{2A}\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}}\xi_\lambda.$$

For the TE mode, $\hat{n}\times\vec{H} = \hat{n}\times\vec{H}_t + \hat{n}\times\hat{z}H_z$ so

$$\left|\hat{n}\times\vec{H}\right|^2 = \left|\hat{n}\times\vec{H}_t\right|^2 + |H_z|^2 = \left(\frac{k}{\gamma_\lambda^2}\right)^2\left|\hat{n}\times\vec{\nabla}_t\psi\right|^2 + |\psi|^2.$$

Again let us write

$$\int_\Gamma \left|\hat{n}\times\vec{\nabla}_t\psi\right|^2 \Big/ \int_A \left|\vec{\nabla}\psi\right|^2 = \frac{C}{A}\xi_\lambda, \qquad \int_\Gamma |\psi|^2 \Big/ \int_A |\psi|^2 = \frac{C}{A}\zeta_\lambda.$$

where ζ_λ is another dimensionless number of order one, and ξ_λ is somewhat differently defined from the TM case, but still of order one. Then

$$\int_\Gamma \left|\hat{n}\times\vec{\nabla}_t\psi\right|^2 \Big/ \int_A |\psi|^2 = \gamma_\lambda^2\frac{C}{A}\xi_\lambda.$$

The attenuation coefficient is

$$\beta_\lambda = \sqrt{\frac{\epsilon}{\mu}}\frac{1}{\sigma\delta_\lambda}\frac{C}{2A}\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}}\left[\xi_\lambda\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)+\zeta_\lambda\left(\frac{\omega_\lambda}{\omega}\right)^2\right]$$

Then for both modes we can write the attenuation coefficient as

$$\beta_\lambda = \sqrt{\frac{\epsilon}{\mu}}\frac{1}{\sigma\delta_\lambda}\frac{C}{2A}\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}}\left[\xi_\lambda+\eta_\lambda\left(\frac{\omega_\lambda}{\omega}\right)^2\right],$$

where $\eta_\lambda = \zeta_\lambda - \xi_\lambda$ for the TE mode. Note that for the TM mode $\eta_\lambda = 0$.

The attenuation coefficient calculated in this approximation diverges as $\omega \to \omega_\lambda$, and is it is proportional to $\sqrt{\omega}$ for large ω, so it has a minimum for some small multiple of ω_λ, $\sqrt{3}$ for the TM modes, but dependent on η_λ/ξ_λ, and hence the shape of the waveguide, for TE modes.

We will skip section 8.6.

1.3 Attenuation for Circular Cylinder

We have seen that the TE and TM modes in a circular wave guide are determined by

$$\psi_{mn}^{\rm TE} = J_m(x'_{mn}\rho/r)\cos m\phi, \qquad \psi_{mn}^{\rm TM} = J_m(x_{mn}\rho/r)\cos m\phi,$$

where x_{mn} and x'_{mn} are the n'th zeros of $J_m(x)$ and $J'_m(x)$ respectively. The cutoff frequencies are given in terms of

$$\gamma_{mn}^{\rm TE} = x'_{mn}/r, \qquad \gamma_{mn}^{\rm TM} = x_{mn}/r.$$

To evaluate the attenuation coeffieients, we need $\int_A \psi^2$, and

$$\int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 = r\int_0^{2\pi} d\phi\gamma^2 J_m'^2(\gamma r)\cos^2\phi = \pi r\gamma^2 J_m'^2(\gamma r)(1+\delta_{m0}),$$

which we need only for TM modes. For TE modes we need

$$\int_\Gamma |\psi|^2 = rJ_m^2(x'_{mn})\int_0^{2\pi}\cos^2 m\phi\, d\phi = \pi r J_m^2(x'_{mn})(1+\delta_{m0}),$$

$$\begin{aligned}\int_\Gamma |\hat{n}\times\nabla_t\psi|^2 &= r\int_0^{2\pi} d\phi\left(\frac{\partial\psi}{r\partial\phi}\right)^2 = \frac{1}{r}J_m^2(x'_{mn})\int_0^{2\pi}(m\sin m\phi)^2\\ &= \frac{\pi}{r}J_m^2(x'_{mn})(1+\delta_{m0}),\end{aligned}$$

The only integral that requires more than table look-up is

$$\int_A \psi^2 \;=\; \int_0^r \rho d\rho J_m^2(\gamma\rho) \int_0^{2\pi} d\phi \cos^2(m\phi) = \pi \int_0^r \rho d\rho J_m^2(\gamma\rho)(1+\delta_{m0}).$$

The integral is connected to the orthnormalization properties of Bessel functions, and is[1]:

$$\int_0^1 \left[J_m\left(x_{mn}u\right)\right]^2 u du = \frac{1}{2} J_{m+1}^2(x_{mn})$$

$$\int_0^1 \left[J_m\left(x'_{mn}u\right)\right]^2 u du = \frac{1}{2}\left(1 - \frac{m^2}{(x'_{mn})^2}\right) J_m^2(x'_{mn})$$

Thus for the TM modes, we have

$$\frac{C}{A}\xi_{mn}^{\text{TM}} = \int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \Big/ (\gamma_{mn}^{\text{TM}})^2 \int_A \psi^2 = \frac{\pi r J_m'^2(x_{mn})}{\frac{\pi r^2}{2} J_{m+1}^2(x_{mn})} = \frac{2}{r}\frac{J_m'^2(x_{mn})}{J_{m+1}^2(x_{mn})}$$

In fact, there is an identity (see footnote again) $J_m'(x) = \frac{m}{x}J_m(x) - J_{m+1}(x)$, which means, as $J_m(x_{mn}) = 0$, that $J_m'(x_{mn}) = -J_{m+1}(x_{mn})$, $\frac{C}{A}\xi_{mn}^{\text{TM}} = 2/r$, and

$$\beta_{mn}^{\text{TM}} = \sqrt{\frac{\epsilon}{\mu}}\frac{1}{r\sigma\delta_\lambda}\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\dfrac{\omega_\lambda^2}{\omega^2}}}$$

for all TM modes.

For the TE modes,

$$\frac{C}{A}\xi_{mn}^{\text{TE}} \;=\; \int_\Gamma |\hat{n}\times\nabla_t\psi|^2 \Big/ (\gamma_{mn}^{\text{TE}})^2 \int_A \psi^2 = \frac{m^2\pi J_m^2(x'_{mn})/r}{\pi(\gamma_{mn}^{\text{TE}})^2 r^2 \frac{1}{2}\left(1-(m/x'_{mn})^2\right) J_m^2(x'_{mn})}$$

$$\;=\; \frac{2m^2}{r(x'^2_{mn}-m^2)}.$$

$$\frac{C}{A}\zeta_{mn}^{\text{TE}} \;=\; \int_\Gamma |\psi|^2 \Big/ \int_A \psi^2 = \frac{\pi r J_m^2(x'_{mn})}{\frac{\pi}{2}\left(1-(m/(x'_{mn})^2)\right) J_m^2(x'_{mn})} = \frac{2x'^2_{mn}}{r\left(x'^2_{mn}-m^2\right)}.$$

So the attenuation coefficient is

$$\beta_{mn}^{\text{TE}} = \sqrt{\frac{\epsilon}{\mu}}\frac{1}{r\sigma\delta_\lambda}\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\dfrac{\omega_\lambda^2}{\omega^2}}}\left[\frac{1}{(x'^2_{mn}-m^2)} + \left(\frac{\omega_\lambda}{\omega}\right)^2\right].$$

[1] Arfken 11.50, problems 11.2.2 and 11.2.3 (3rd Ed.) or see `http://www.physics.rutgers.edu/grad/504/lects/bessel.pdf`.

For TM modes, $\omega_{mn}^{\text{TM}} = x_{mn}c/r$, where $c = 1/\sqrt{\mu\epsilon}$ is the speed of light, For copper, the resistivity is $\rho = \sigma^{-1} = 1.7\times 10^{-8}\ \Omega\cdot\text{m}$, and we may take the permeability to be essentially μ_0. Also $\omega_\lambda = \gamma_\lambda c$. $\delta_\lambda = \sqrt{2/\mu_c\sigma\omega_\lambda}$ $\epsilon_0 = 8.854\times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2$, so

$$\sqrt{\frac{\epsilon}{\mu}}\frac{1}{\sigma\delta_\lambda} = \sqrt{\frac{c\epsilon_0\gamma_\lambda}{2\sigma}} = 4.75\times10^{-6}\sqrt{\gamma_\lambda}\sqrt{\frac{\text{m}}{\text{s}}\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\Omega\text{m}} = 4.75\times10^{-6}\ \text{m}^{1/2}\cdot\sqrt{\frac{x_{mn}}{r}}.$$

The units combine to $\text{m}^{1/2}$ as $1\ \Omega = 1\text{V/A} = 1(\text{J/C})/(\text{C/s}) = \text{Nms/C}^2$.

In comparison to the TM_{12} mode for a square of side a, we see that $\beta^{\text{TM}} = \frac{a}{2r}\beta_{12}^{\square\text{TM}}$. As the cutoff frequencies are $2.4048c/r$ and $\sqrt{5}\pi c/a$ respectively, we see that the comparable dimensions are $r = (2.4048/\sqrt{5}\pi)a = 0.342a$, much smaller, and then $a/2r = 1.46$, so the smaller pipe does have faster attenuation.

For TE modes, there is an extra factor of

$$\frac{1}{(x'^2_{mn}-m^2)} + \left(\frac{\omega_\lambda}{\omega}\right)^2.$$

which for the lowest mode is $0.4185 + (\omega_\lambda/\omega)^2$ compared to $0.5 + (\omega_\lambda/\omega)^2$ for the square. But the cutoff frequencies are now $1.841c/r$ and $\sqrt{2}\pi c/a$, so comparable dimensions have $r = 1.841a/\sqrt{2}\pi = 0.414a$.

1.4 Resonant Cavities

We have considered wave guides uniformly extended in the z direction, infinite in both directions, and found that there are modes λ of propagation with arbitrary definite wavenumber k and frequency ω given by $\mu\epsilon\omega^2 = k^2 + \gamma_\lambda^2$. Thus for a particular λ and $\omega > \omega_\lambda$, there are two possible waves, a right-moving and a left-moving one. Superposition will then give us standing waves suitable to describe a resonant cavity made by placing flat conducting end-caps on the wave guide, say at $z = 0$ and $z = d$.

Thus quite generally each field will be a superposition of wave with $k = |k|$ and one with $k = -|k|$. For the TM case, the determining field is

$$E_z = \left(\psi^{(k)}e^{ikz} + \psi^{(-k)}e^{-ikz}\right)e^{-i\omega t},$$

In calculating the transverse field, the piece coming from $\psi^{(-k)}$ needs the minus in 8.33, so

$$\vec{E}_t = i\frac{k}{\gamma_\lambda^2}\left(\vec{\nabla}_t\psi^{(k)}e^{ikz} - \vec{\nabla}_t\psi^{(-k)}e^{-ikz}\right)$$

This is a field parallel to the conductor surface at $z = 0$ and $z = d$, so must vanish (or be very small) there for a perfect (good) conductor endcap. Vanishing of $\vec{E}_t$ at $z = 0$ implies $\psi^{(k)} = \psi^{(-k)}$, and then $\psi^{(k)}(2i \sin kd) = 0$ from $z = d$. As we don't want $\psi = 0$, we see that $k = p\pi/d$ with p an integer,

$$\left.\begin{aligned} E_z &= \cos\left(\frac{p\pi z}{d}\right)\,\psi(x,y) \\ \vec{E}_t &= -\frac{p\pi}{d\gamma_\lambda^2}\sin\left(\frac{p\pi z}{d}\right)\vec{\nabla}_t\psi \\ \vec{H}_t &= i\frac{\epsilon\omega}{\gamma_\lambda^2}\cos\left(\frac{p\pi z}{d}\right)\hat{z}\times\vec{\nabla}_t\psi \end{aligned}\right\} \qquad \begin{cases}\text{for TM modes} \\ \quad\text{with } p \in \mathbb{Z}\end{cases}$$

where in using 8.26, we recall that k takes a minus sign for the part of the field flowing leftward.

For TE modes, the determining field is H_z, which must vanish at $z = 0$ and $z = d$ if the endcaps are perfect conductors and exclude magnetic fields, as $\hat{n} \cdot \vec{B}$ is continuous at the boundary. Thus

$$\left.\begin{aligned} H_z &= \sin\left(\frac{p\pi z}{d}\right)\,\psi(x,y) \\ \vec{H}_t &= \frac{p\pi}{d\gamma_\lambda^2}\cos\left(\frac{p\pi z}{d}\right)\vec{\nabla}_t\psi \\ \vec{E}_t &= -i\frac{\omega\mu}{\gamma_\lambda^2}\sin\left(\frac{p\pi z}{d}\right)\hat{z}\times\vec{\nabla}_t\psi \end{aligned}\right\} \qquad \begin{cases}\quad\text{for TE modes} \\ \text{with } p \in \mathbb{Z}, p \neq 0.\end{cases}$$

As for the waveguide, the values of γ_λ depend on the mode (TE or TM) and the cross section, which often means two indices. For example, for a circular guide, there is an angular index m and another index n specifying which root of J_m (for TM) or of $dJ(x)/dx$ (for TE). With x_{mn} the n'th zero of $J_m(x)$ (not counting $x = 0$) and x'_{mn} the n'th zero of $\frac{dJ}{dx}(x)$, we have $\gamma_{mn} = x_{mn}/R$ (TM modes) or $\gamma_{mn} = x'_{mn}/R$ (TE modes). As $\mu\epsilon\omega^2 = k^2 + \gamma^2$ we have

$$\begin{aligned} \omega_{mnp} &= \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\frac{x_{mn}^2}{R^2} + \frac{p^2\pi^2}{d^2}} \qquad \text{with } p \geq 0 \text{ for TM modes,} \\ \omega_{mnp} &= \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\frac{x'^2_{mn}}{R^2} + \frac{p^2\pi^2}{d^2}} \qquad \text{with } p > 0 \text{ for TE modes.} \end{aligned}$$

The lowest TM mode is thus $\omega_{010} = x_{01}/\sqrt{\mu\epsilon}\,R = 2.405/\sqrt{\mu\epsilon}\,R$, which is independent of the length d of the cavity. As p cannot be zero for TE modes, (or the determining field $H_z = 0$) the lowest TE mode is

$$\omega_{111} = \frac{1.841}{\sqrt{\mu\epsilon}R}\sqrt{1 + 2.912\frac{R^2}{d^2}},$$

$(\pi/1.841 = 2.912)$. This mode has the advantage that its frequency can be tuned by moving a piston back and forth, changing d.

This calculation has assumed no power losses, but of course a real cavity will generally have walls of finite conductivity, and power will be lost as we discussed earlier in the walls, not only along the z direction but also in the endcaps. Again the power lost will be proportional to the energy stored in the cavity. Let Q be the 2π times the energy stored U divided by the energy lost in one cycle ΔU (in time $dt = 2\pi/\omega$), so $Q = 2\pi\frac{U}{\Delta U}$. Assuming $Q \gg 1$, the energy loss per cycle will be small compared to U, with $\Delta U \approx -\frac{2\pi}{\omega}\frac{dU}{dt}$, and the energy will decay exponentially, with

$$U(t) = U(0)e^{-\omega t/Q}, \quad Q = \omega U/|dU/dt|.$$

This means that if at time zero something excites an electromagnetic field in the cavity, the fields will have a time dependence[2]

$$E(t) = E_0 e^{-i\omega_0(1-i/2Q)t}\Theta(t),$$

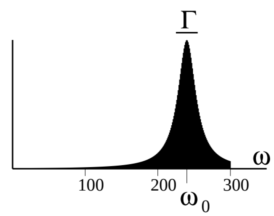
where $\Theta(t) = 1$ for $t \geq 0$ and zero earlier. Thus the frequency response to what is essentially a delta-function excitation (and therefore equal for all frequencies) is

$$\begin{aligned} E(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} E(t)e^{i\omega t}dt = \frac{1}{\sqrt{2\pi}}E_0\int_0^\infty e^{i(\omega-\omega_0-i\Gamma/2)t}dt \\ &= \frac{iE_0}{\sqrt{2\pi}}\frac{1}{\omega-\omega_0-i\Gamma/2}, \end{aligned}$$

[2]My ω_0 is what Jackson calls $\omega_0 + \Delta\omega$, with his ω_0 the resonant frequency of the undamped cavity. The change in resonant frequency due to damping is generally small, a fractional change of the order $1/Q$, and I will ignore that effect.

where $\Gamma := \omega_0/Q$. This response determines how the cavity will respond to excitations of any frequency, with the energy absorbed proportional to

$$|E(\omega)|^2 \propto \frac{1}{(\omega-\omega_0)^2+\Gamma^2/4}.$$



This resonance shape is the form of response the simplest resonant structures have in response to a stimulus of frequency ω, as for example the ratio of the energy of damped harmonic oscillator to the driving force, or the resonant absorption of light by an atomic transition. In nuclear physics this is called the Breit-Wigner amplitude. Γ, mistakenly called the half-width, is actually the full width of the region with a response at least half the maximum value, which is $\omega \in [\omega_0 - \Gamma/2, \omega_0 + \Gamma/2]$.

The value of Γ for a resonant cavity can be calculated as for the attenuation of a waveguide. That is, we compare the power lost in the walls to the energy in the electromagnetic fields in the cavity. This is done in Jackson, pp 373-374, based on the same tools as used in calculating the attenuation of the waveguide, but I will skip it.

2 Earth and Ionosphere

An interesting resonant cavity is formed by the surface of the Earth and the ionosphere, a layer of ionized gas starting about 100 km up, which provides sufficient conductivity to reflect radio waves. But this cavity is not a cylinder with endcaps, but clearly calls out for spherical coordinates. As you also need an introduction to other curvilinear coordinate systems to do your homework, and as you have told me you have not learned about these, let us digress to discuss curvilinear coordinates in general, orthogonal coordinates more particularly, and finally spherical coordinates.