Physics 504, Lectures 6
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1 Lecture 6

1.1 Power Loss at Interface

Last time we found the fields die off with skin depth

$$\delta = \sqrt{\frac{2}{\mu_c \omega \sigma}}$$

with

$$\begin{aligned} \vec{H}_c &= \vec{H}_\parallel e^{-\xi/\delta} e^{i\xi/\delta}, \\ \vec{E}_c &= \sqrt{\frac{\mu_c \omega}{2\sigma}} (1-i) \hat{n} \times \vec{H}_\parallel e^{-\xi/\delta} e^{i\xi/\delta}, \end{aligned}$$

where $\vec{H}_\parallel$ is the tangential magnetic field at the surface outside the conductor, and

$$\vec{E}_\parallel = \sqrt{\frac{\mu_c \omega}{2\sigma}} (1-i) \hat{n} \times \vec{H}_\parallel.$$

There are two ways to calculate the power. First, the flow of energy through the surface is given by the Poynting vector $\vec{S} = \vec{E} \times \vec{H}$. Because we are using complex fields E and $H \propto e^{-i\omega t}$, of which only the real parts are physical, we need $\langle \vec{S} \rangle = \frac{1}{2}$ Re $\vec{E} \times \vec{H}^*$. So the power loss per unit area is

$$\begin{aligned} \frac{dP_{\rm loss}}{dA} &= -\hat{n} \cdot \langle \vec{S} \rangle = -\frac{1}{2} \sqrt{\frac{\mu_c \omega}{2\sigma}} \hat{n} \cdot \ {\rm Re}\ \left[(1-i)(\hat{n} \times \vec{H}_\parallel) \times \vec{H}_\parallel^* \right] \\ &= \frac{\mu_c \omega \delta}{4} |\vec{H}_\parallel|^2 = \frac{1}{2\sigma\delta} |\vec{H}_\parallel|^2 \end{aligned}$$

The other way is to ask about the ohmic losses, with power lost per unit volume of $\frac{1}{2} \vec{J} \cdot \vec{E}^* = |\vec{J}|^2 / 2\sigma$. As $|\vec{J}| = \sigma \vec{E}_c = \frac{\sqrt{2}}{\delta} |\vec{H}_\parallel| e^{-\xi/\delta}$, the power loss per unit area is

$$\frac{dP_{\rm loss}}{dA} = = \frac{1}{\delta^2 \sigma} |\vec{H}_\parallel|^2 \int_0^\infty d\xi \, e^{-2\xi/\delta} = \frac{1}{2\delta\sigma} |\vec{H}_\parallel|^2.$$

We can also express this in terms of the surface current, where we mean the total current near the surface,

$$\vec{K}_{\rm eff} = \int_0^\infty d\xi\, \vec{J}(\xi) = \frac{1}{\delta}\hat{n}\times\vec{H}_\parallel \int_0^\infty d\xi\,(1-i)e^{-\xi(1-i)/\delta} = \hat{n}\times\vec{H}_\parallel.$$

Thus

$$\frac{dP_{\rm loss}}{dA} = \frac{1}{2\sigma\delta}|\vec{K}_{\rm eff}|^2.$$

Thus we may view $1/\sigma\delta$ as the surface resistance, or the ratio $\vec{E}_\parallel/\vec{K}_{\rm eff} = (1-i)/\sigma\delta$ as the surface impediance Z.

1.2 Waveguides

As our situation involves time-independent boundary conditions and linear equations, we can use a fourier transform in time, with

$$\begin{aligned}\vec{E}(\vec{x},t) &= \vec{E}(x,y,z)\; e^{-i\omega t}\\ \vec{B}(\vec{x},t) &= \vec{B}(x,y,z)\; e^{-i\omega t}\end{aligned}$$

with the understanding that the physical fields are the real part of these expressions, and of course we could have superpositions of different frequencies, but these don't interact.

In the interior $\rho = 0$, $\vec{J} = 0$, so

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = i\omega\vec{B}, \qquad \vec{\nabla}\cdot\vec{E} = 0, \qquad \vec{\nabla}\cdot\vec{B} = 0,$$

$$\vec{\nabla}\times\vec{B} = \mu\vec{\nabla}\times\vec{H} = \mu\frac{\partial\vec{D}}{\partial t} = \mu\epsilon\frac{\partial\vec{E}}{\partial t} = -i\omega\mu\epsilon\vec{E}.$$

Then

$$\nabla^2\vec{E} = -\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) = -\vec{\nabla}\times(i\omega\vec{B}) = -\omega^2\mu\epsilon\vec{E}.$$

and similarly

$$\left(\nabla^2 + \omega^2\mu\epsilon\right)\vec{B} = 0. \tag{1}$$

Let us assume our problem involves a cylinder of arbitrary cross-section, but uniform in z (though possibly only on an interval in z, possibly capped at the ends). Then we can also fourier transform in z,

$$\vec{E}(x,y,z,t) = \vec{E}(x,y)e^{ikz-i\omega t}, \qquad \vec{B}(x,y,z,t) = \vec{B}(x,y)e^{ikz-i\omega t},$$

where k could take either sign, and we might take a superposition if we need to. Then the Helmholtz equation (1) for $\vec{B}$ and $\vec{E}$ give

$$\left[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)\right] \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} = 0, \quad \nabla_t^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Break down the vectors into transverse and longitudinal parts:

$$\vec{E} = E_z \hat{z} + \vec{E}_t, \quad \vec{B} = B_z \hat{z} + \vec{B}_t, \quad \text{with } \vec{E}_t \perp \hat{z},\ \vec{B}_t \perp \hat{z}.$$

Now

$$(\vec{\nabla} \times \vec{E})_z = (\vec{\nabla}_t \times \vec{E}_t)_z = i\omega B_z, \tag{2}$$

$$(\vec{\nabla} \times \vec{E})_\perp = \hat{z} \times \frac{\partial \vec{E}_t}{\partial z} - \hat{z} \times \nabla_t E_z = i\omega \vec{B}_t. \tag{3}$$

For any vector $\vec{V}$, $\hat{z} \times (\hat{z} \times \vec{V}) = -\vec{V} + \hat{z}(\hat{z} \cdot V)$, so for a transverse vector $\hat{z} \times (\hat{z} \times \vec{V}_t) = -\vec{V}_t$. Taking $\hat{z}\times$ Eq. (3) gives

$$\frac{\partial \vec{E}_t}{\partial z} - \vec{\nabla}_t E_z = -i\omega \hat{z} \times \vec{B}_t. \tag{4}$$

The same decomposition of $\vec{\nabla} \times \vec{B} = -i\omega\mu\epsilon\vec{E}$ gives

$$\left(\vec{\nabla}_t \times \vec{B}_t\right)_z = -i\omega\mu\epsilon E_z \tag{5}$$

$$\frac{\partial \vec{B}_t}{\partial z} - \vec{\nabla}_t B_z = i\omega\mu\epsilon \hat{z} \times \vec{E}_t. \tag{6}$$

Of course the divergencelessness of $\vec{E}$ and $\vec{B}$ give

$$\vec{\nabla}_t \cdot \vec{E}_t + \frac{\partial E_z}{\partial z} = 0, \qquad \vec{\nabla}_t \cdot \vec{B}_t + \frac{\partial B_z}{\partial z} = 0.$$

Making use of the fourier transform in z, we have

$$ik\vec{E}_t + i\omega \hat{z} \times \vec{B}_t = \vec{\nabla}_t E_z \tag{7}$$

$$ik\vec{B}_t - i\omega\mu\epsilon \hat{z} \times \vec{E}_t = \vec{\nabla}_t B_z \tag{8}$$

Solving 8 for $\vec{B}_t$ and plugging into 7, and then the reverse for $\vec{E}_t$, gives

$$E_t = i\frac{k\vec{\nabla}_t E_z - \omega \hat{z} \times \vec{\nabla}_t B_z}{\omega^2\mu\epsilon - k^2} \tag{9}$$

$$B_t = i\frac{k\vec{\nabla}_t B_z + \omega\mu\epsilon \hat{z} \times \vec{\nabla}_t E_z}{\omega^2\mu\epsilon - k^2} \tag{10}$$

Thus E_z and B_z determine the rest, unless $k^2 = k_0^2 := \mu\epsilon\omega^2$, in which case both E_z and B_z are zero. Then there are no longitudinal fields, and we call this a transverse electromagnetic (TEM) wave. It travels in the z direction at the speed $1/\sqrt{\mu\epsilon}$ which we would have for a plane wave in an infinite medium, and with the wave number $k = k_0 := \omega\sqrt{\mu\epsilon}$ that the wave would have in an infinite medium. These TEM fields satisfy $\vec{\nabla}_t \cdot \vec{E}_t = 0$, $\vec{\nabla}_t \times \vec{E}_t = i\omega B_z = 0$, and therefore $\vec{E}_t = -\vec{\nabla}_t\Phi$ for some (not necessarily singlevalued) function Φ on the cross section, with $\nabla^2\Phi = 0$. As $\vec{E}_\parallel = 0$ at the boundary, each boundary is an equipotential of Φ, and if the cross section is simply connected, the only solution is $\Phi =$ constant, $\vec{E}_t = 0$. Thus there can be no TEM wave on a simply connected cylinder, but the TEM is the principal wave on a coaxial cable (which has an inner and an outer conductor, with different Φ, or for parallel wires, as in an old 300 Ω television cable. Note that if μ and ϵ are nondispersive, so is the TEM wave, with no cutoff on the transmission frequency or wavelength.

For perfectly conducting waveguides we saw that at the boundary $\vec{n}\times\vec{E} = 0$, $\vec{n}\cdot\vec{B} = 0$. This means $E_z = 0$ and $\vec{E}_t \parallel \vec{n}$ at the boundary. From $\vec{n}\cdot(\partial\vec{B}_z/\partial z - i\mu\epsilon\omega\hat{z}\times\vec{E}_t - \vec{\nabla}_t B_z) = 0$, the first two terms vanish, the second because $\vec{E}_t \parallel \hat{n}$ at the boundary, so $\partial B_z/\partial n|_S = 0$. Thus E_z satisfies a Dirichlet zero condition and B_z satisfies a Neumann zero condition boundary conditions in two dimensions. For simply connected cross section, there are in general no nonzero solutions, except for certain discrete values of the constant $\mu\epsilon\omega^2 - k^2$, and the allowed values will, in general, be different for the two possibilities. So in general if there is a solution for one condition, say $E_z = 0$ on the boundary, we will have $B_z \equiv 0$, the magnetic field is purely transverse, and we call this a transverse magnetic (TM) mode. For the other condition, $\vec{n}\cdot\vec{\nabla}_t B_z = 0$ on the boundary, we have $E_z \equiv 0$, $\vec{E}$ is purely transverse, and this is called a transverse electric (TE) mode.

1.3 Waveguide impediance, modes, and cutoff frequencies

Note that for a TM mode with vanishing B_z, (9) and (10) give

$$\text{TM:}\qquad (k_0^2 - k^2)\vec{E}_t = ik\vec{\nabla}_t E_z, \qquad (k_0^2 - k^2)\vec{B}_t = i\mu\epsilon\omega\hat{z}\times\vec{\nabla}_t E_z$$

or $\vec{H}_t = \epsilon\omega k^{-1}\hat{z}\times\vec{E}_t$, while for a TE mode with vanishing E_z

$$\text{TE:}\qquad (k_0^2 - k^2)\vec{E}_t = -i\omega\hat{z}\times\vec{\nabla}_t B_z, \qquad (k_0^2 - k^2)\vec{B}_t = ik\vec{\nabla}_t B_z,$$

so $\vec{E}_t = -\omega\hat{z} \times \vec{B}_t/k$. Premultiplying by $\hat{z}\times$, we have $H_t = k\hat{z} \times E_t/\mu\omega$. In both cases we have

$$\vec{H}_t = \frac{1}{Z}\hat{z} \times \vec{E}_t, \qquad Z = \begin{cases} \frac{k}{\epsilon\omega} = \frac{k}{k_0}\sqrt{\frac{\mu}{\epsilon}} & \text{TM} \\ \frac{\mu\omega}{k} = \frac{k_0}{k}\sqrt{\frac{\mu}{\epsilon}} & \text{TE} \end{cases}$$

Now each component of $\vec{E}$ and $\vec{B}$ is of the form $\Psi(x,y)e^{ikz-i\omega t}$ where each Ψ satisfies

$$\left(\nabla_t^2 + \gamma^2\right)\Psi = 0 \qquad \gamma^2 = \mu\epsilon\omega^2 - k^2$$

For the TM and TE modes they are determined by a single scalar ψ, and for $t = z = 0$ are given by

$$\begin{array}{llll} \text{TM:} & E_z = \psi, & \vec{E}_t = ik\gamma^{-2}\vec{\nabla}_t\psi & \psi|_\Gamma = 0 \\ \text{TE:} & H_z = \psi, & \vec{H}_t = ik\gamma^{-2}\vec{\nabla}_t\psi & \hat{n}\cdot\vec{\nabla}_t\psi|_\Gamma = 0, \end{array}$$

with $(\nabla_t^2 + \gamma^2)\,\psi = 0$. Note that with these conditions,

$$0 = \int_A \psi^*\left(\nabla_t^2 + \gamma^2\right)\psi = \int_A \vec{\nabla}_t\cdot(\psi^*\vec{\nabla}_t\psi) - \int_A (\vec{\nabla}_t\psi)^*\cdot\vec{\nabla}_t\psi + \gamma^2\int_A |\psi|^2,$$

where A is the cross section. The first integral is a divergence, so is $\oint_{\partial A}\psi^*\vec{n}\cdot\vec{\nabla}_t\psi$, which vanishes from either boundary condition, the second integral is strictly positive unless ψ is a constant[1], and the coefficient of γ^2 is positive, so γ^2 is positive. There will be solutions of the two-dimensional Helmholtz equation for discrete positive values γ_λ^2. For each frequency ω, there can be waves with wave numbers

$$k_\lambda^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2,$$

so only waves with $\omega > \omega_\lambda := \gamma_\lambda/\sqrt{\mu\epsilon}$ can propagate. With $k_\lambda^2 < 0$ we can have cutoff modes (or evanescent modes) which do note propagate but decay with z. Note $k_\lambda < \sqrt{\mu\epsilon}\,\omega$, the value the wavelength would have in an infinite medium, so the wavelength in the waveguide is longer than in $\mathbb{R}^3$. The phase velocity $v_p = \omega/k_\lambda > 1/\sqrt{\mu\epsilon}$, greater than in $\mathbb{R}^3$.

[1] In which case we must have a TE mode, but then E_t and B_t are both zero, $\vec{E} = 0$, and thus $B_z =$ constant.

1.4 An Example

We see that finding the dispersion of a cylindrical waveguide involves solving the two dimensional Helmholtz equation with boundary conditions specified on Γ, the cross section's intersection with the surface.

$$\left(\nabla_t^2 + \gamma^2\right)\psi = 0 \quad \text{with} \quad \psi|_\Gamma = 0 \text{ (TM)} \quad \text{or } \hat{n}\cdot\vec{\nabla}\psi|_\Gamma = 0 \text{ (TE)}.$$

There are a number of coordinate systems for which the Laplacian operator can be separated, and if the boundary shapes are suitable, it is straightforward to find solutions. Of course the simplest is a rectangular wave guide, for which we can use cartesian coordinates. This is worked out in Jackson, section 8.4, and you should definitely work through it (rectangular waveguides have appeared on the qualifier!), but it is quite clear and it would add nothing for me to repeat the solution, so instead, lets consider a circular cylindrical waveguide of radius r.

Naturally we should use cylindrical coordinates, or for the cross section simply polar coordinates ρ, ϕ. The Laplace operator in polar coordinates is

$$\nabla_t^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}.$$

If we make an ansatz that the solution $\psi(\rho,\phi) = R(\rho)\Phi(\phi)$, we have

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2 R(\rho)\right)\Phi(\phi) + \frac{1}{\rho^2}R(\rho)\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = 0.$$

Dividing by $R(\rho)\Phi(\phi)$ and multiplying by ρ^2 gives

$$\begin{aligned} \frac{1}{R(\rho)}\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2\rho^2 R(\rho)\right) & \\ + \frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} &= 0. \end{aligned}$$

The first line depends on ρ but not on ϕ, which the second depends on ϕ but not on ρ, so they must be equal and opposite constants,

$$\begin{aligned} \frac{1}{R(\rho)}\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2\rho^2 R(\rho)\right) &= C \\ \frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} &= -C. \end{aligned}$$

The second equation,

$$\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} + C\Phi(\phi) = 0$$

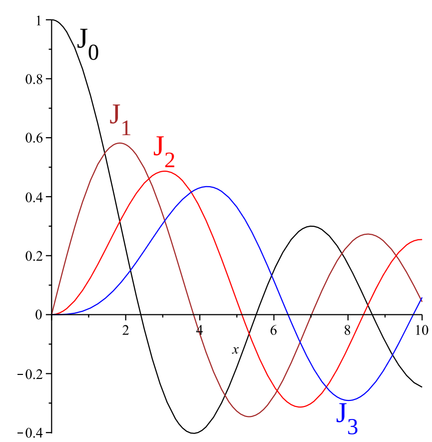
has the solution $\Phi(\phi) = e^{\pm i\sqrt{C}\phi}$. As we need a solution periodic in ϕ, that is $\Phi(\phi + 2\pi) = \Phi(\phi)$, we see that $\sqrt{C}$ must be an integer, m. Then the first equation is

$$\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \gamma^2\rho^2 - m^2\right) R(\rho) = 0,$$

which is the Bessel equation, with solutions regular at the origin given by $J_m(\gamma\rho)$.

It is straightforward to satisfy the boundary condition by demanding that γr is a zero of J_m (for TM waves) or of $dJ_m(x)/dx$ (for TE waves). These can be looked up in many books[2].

We have $x^{\rm TM}_{mn}$ the n'th zero of J_m and $x^{\rm TE}_{mn}$ the n'th zero of J'_m. In terms of that numerical value, x_{mn}, we have $\gamma = x_{mn}/r$, and the tube can only support electromagnetic waves with a frequency greater than the cutoff frequency $\omega_{mn} = x_{mn}/r\sqrt{\mu\epsilon}$. The smallest of these roots is that J'_1, with $x^{\rm TE}_{11} = 1.8412$, and the next is that of J_0, with $x^{\rm TM}_{01} = 2.4048$. If the waveguide is 5 cm in diameter, and filled with air $\sim$ vacuum, this gives a cutoff on TE modes of $f = \frac{\omega}{2\pi} = 3.5$ GHz and 4.6 GHz for the lowest TM mode.



[2]For example, Arfken III p. 581, or Jackson p. 114 and 370, or, for far more, Abramowitz and Stegun, p. 409 and 411.