Physics 504, Lecture 15
March 25, 2010

We now begin with Chapter 12, on the relativistic dynamics of charged particles in interaction with electromagnetic fields. I am permuting the order in Jackson and will consider sections 2-4 before section 1.

1 Motion of charged particles in fixed external fields

There are many applications which require an analysis of how charged particles move in static external (macroscopic) electromagnetic fields. These include the bending of beams to make circular accelerators for nuclear and particle physics, understanding plasmas in deep space and in attempts to make fusion energy devices, designing velocity and momentum separators for beams of particles, understanding van Allen belts which cause the auroras, and many other applications. Of course, all of these analyses are based on the general formula $dp^\alpha/d\tau = (q/c)F^{\alpha\beta}U_\beta$, or in nonrelativistic language,

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right), \quad \frac{dE}{dt} = q\vec{v}\cdot\vec{E}.$$

1.1 Constant Uniform $\vec{B}$ Only

First let us consider a uniform constant $\vec{B}$ with no electric field. The energy is constant, so so are $|v|$ and γ, and we have

$$\frac{d\vec{v}}{dt} = \frac{1}{\gamma m}\frac{d\vec{p}}{dt} = \frac{q}{\gamma mc}\vec{v}\times\vec{B} = \vec{v}\times\vec{\omega}_B, \quad \text{where } \vec{\omega}_B = \frac{q}{\gamma mc}\vec{B} = \frac{qc\vec{B}}{E}.$$

Thus the component of $\vec{v}$ parallel to $\vec{B}$ is a constant, and the other two components rotate counterclockwise about the $\vec{B}$ direction if the charge is positive. The position component parallel to $\vec{B}$ grows linearly with time, while the motion transverse to that is in a circle with angular velocity ω_B. The radius a of this circle is determined from $v_\perp = \omega_B a$, so

$$a = \frac{v_\perp}{\omega_B} = \frac{p_\perp}{m\gamma}\Big/\frac{qB}{\gamma mc} = \frac{p_\perp c}{qB}.$$

This can be used to determine the momentum of a particle by measuring the radius of curvature in a magnetic field, and has been used in particle detectors at all high energy accelerators ever since the field began. It is also the formula which tells us how strong the magnetics must be at the LHC at CERN to get 7 TeV protons to bend in a circle with a circumference of only 27 km.[1]

The fact that the frequency of revolution $\omega_B/2\pi = qB/2\pi mc\gamma$ is nearly constant independent of v as long as the particle is nonrelativistic ($\gamma \approx 1$) makes possible the continuous acceleration of particles in a cyclotron[2], which consists of two "D" shaped cavities with an oscillating voltage between them, with a constant magnetic field normal to their plane which provides the bending to make particles go in semicircles of ever larger radius, each in half a period of the oscillating voltage, until they reach the outside of the cyclotron and come out in a "high energy" beam. The first cyclotron was built by Lawrence (not Lorentz) in 1929, was 4 inches in diameter. Three years later an 11 inch one set the high energy record for protons at 1 MeV. Lawrence kept building bigger and bigger machines, ignoring warnings that $\gamma = 1$ was not an exact statement, so to get the 184 inch cyclotron to work, particles had to be accelerated in bunches with the frequency gradually decreased in synchonicity with the increasing particle energies and γ's.

1.2 Constant Uniform $\vec{E}$ and $\vec{B}$

Next. let's consider adding a fixed uniform electric field to our magnetic one. The electric field does do work on the particle, so we can no longer assume $|v|$ and γ are constant, and the situation is considerably more complicated. If the electric field is perpendicular to the magnetic field, however, we can use a Lorentz transformation to a more suitable frame to help. If we take $\vec{E}$ to be in the y direction and $\vec{B}$ in the z, we can apply a Lorentz transformation in the x direction, with $u_x = c\tanh\zeta$. Then the Lorentz transformation $A^\mu{}_\nu$

[1] With $B = P_\perp c/qR$ in gaussian units, but $B = P_\perp/qR$ in SI units. As $P_\perp \approx E/c$ and $E/q = 7\times 10^{12}$ V, $R = 4300$ m, $B = 5.4$ T. Unfortunately the 1232 dipole magnets, each 14.3 m long, do not cover the whole circumference, but only 17.6 km, so the magnets need to be 8.3 T, which is considerably harder to maintain.

[2] A nice web page is `http://www.aip.org/history/lawrence/epa.htm`

and the primed and unprimed $F^{\mu\nu}$ are

$$A^\mu{}_\nu = \begin{pmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & -E_y & 0 \\ 0 & 0 & -B_z & 0 \\ E_y & B_z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow F'^{\mu\nu} = \begin{pmatrix} 0 & 0 & -E'_y & 0 \\ 0 & 0 & -B'_z & 0 \\ E'_y & B'_z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with

$$\begin{aligned} E'_y &= \cosh\zeta\, E_y - \sinh\zeta\, B_z \\ B'_z &= \cosh\zeta\, B_z - \sinh\zeta\, E_y \end{aligned}$$

or, more generally, as long as $\vec{u} \perp \vec{B}$ and $\vec{u} \perp \vec{E}$,

$$\vec{E}' = \gamma(\vec{E} + \frac{\vec{u}}{c} \times \vec{B}), \qquad \vec{B}' = \gamma(\vec{B} - \frac{\vec{u}}{c} \times \vec{E}).$$

If we choose $\vec{u} = c\vec{E} \times \vec{B}/B^2$, we have

$$\begin{aligned} \vec{E}' &= \gamma\left(\vec{E} + (\vec{E} \times \hat{B}) \times \hat{B}\right) = \gamma\left(\vec{E} - \vec{E} + (\vec{E} \cdot \hat{B})\hat{B}\right) = 0 \\ \vec{B}' &= \gamma\left(\vec{B} - \frac{1}{B^2}(\vec{E} \times \vec{B}) \times \vec{E}\right) = \gamma\vec{B}\left(1 - \frac{E^2}{B^2}\right) = \frac{1}{\gamma}\vec{B}, \end{aligned}$$

as $\vec{u}^2/c^2 = E^2/B^2$. So in the $\mathcal{O}'$ frame, we have our previous situation: the particle spirals around the $\vec{B}'$ field, though more slowly than before[3]. But in the original $\mathcal{O}$ frame, there is an additional "$\vec{E} \times \vec{B}$ drift" velocity $\vec{u} = c\vec{E} \times \vec{B}/B^2$. Note that this velocity is in the same direction regardless of the sign of the charge of the particle, as it depends only on the fields $\vec{E}$ and $\vec{B}$, while the helical motion is reversed for particles of opposite charge.

There is an important special case, when the helical motion degenerates into constant motion along the $\vec{B}'$ field, so $\vec{v}'$ is a constant in the $\vec{B}$ direction, but more importantly the drift velocity in the $\vec{E} \times \vec{B}$ direction is $u = cE/B$.

[3]Note that the γ here, $\gamma(E/B)$, is not the same as the particle's γ which enters into the expression for ω.

Only particles with this value of the velocity component in that direction will travel in straight lines. and so a series of apertures can serve as a velocity selector. You learned all this as freshman, though then you assumed $\vec{u} \perp \vec{B}$, which we now see is not required.

Transforming to a frame moving at cE/B is not possible if $E > B$. Naïve use of the Lorentz transformation will give imaginary coordinates and fields. But there is a value of u which will annihilate the $\vec{B}$ field instead of the $\vec{E}$ field, so we choose $\vec{u} = c\vec{E} \times \vec{B}/E^2$ and we are left with a problem with a constant uniform $\vec{E}'$, no $\vec{B}'$, and a constant $d\vec{p}'/dt'$. Nonrelativistically this would give simple ballistic (parabolic) motion. Relativistically there are corrections due to the variation of γ in $\vec{p} = m\gamma\vec{v}$, so the motion is more complicated, but still analytically solvable (see Problem 12.3).

As you will see from homework, $E^2 - B^2$ and $\vec{E} \cdot \vec{B}$ are both invariants. This is why, for $\vec{E} \perp \vec{B}$, we had two distinct cases, depending on the sign of $E^2 - B^2$. This also shows that if $\vec{E}$ and $\vec{B}$ are not perpendicular in any frame, so $\vec{E} \cdot \vec{B} \neq 0$ in that frame, the same is true in any frame, and it is not possible to do a Lorentz transformation to a frame in which one of them vanishes. Still, the problem of uniform static $\vec{E}$ and $\vec{B}$ is solvable by brute force using cartesian coordinates.

1.3 Static Non-Uniform Magnetic Fields

We have seen that in a uniform magnetic field particles spiral around magnetic field lines, with a constant velocity component along the line. The radius of the helical motion sets a scale so that it makes sense to say that the field, rather than being absolutely uniform, is varying slowly (in space) on the scale of the radius. This will prove to be an important approximation.

First consider a field with constant direction and uniform in that direction, but varying slowly in a perpendicular direction. Say B_z has a gradient in the x direction. As there is no electric field, the particle has a constant speed and γ, and as the force is transverse to z, a constant v_z. Let the motion in the x-y plane that the particle would have had if there were no gradient be given by position $\vec{x}_\perp(t)$ and velocity $\vec{v}_0(t)$, with $\vec{x}_\perp(t)$ measured from the axis of the helix. The total $\vec{v}_\perp(t) = \vec{v}_0(t) + \vec{v}_1(t)$. Let us work to first order in the gradient of B, which is also the order of v_1. The equation of motion is

$$\frac{d\vec{v}_\perp}{dt} = \frac{q}{\gamma mc}\vec{v}_\perp \times \vec{B}(\vec{x}) \approx \frac{q}{\gamma mc}\vec{v}_\perp \times \vec{B}_0 \left(1 + \frac{1}{B_0}\vec{x}_\perp \cdot \vec{\nabla}_\perp B\Big|_0\right).$$

Then

$$\frac{d(\vec{v}_0 + \vec{v}_1)}{dt} = \frac{q}{\gamma mc}\vec{v}_0 \times \vec{B}_0 + \frac{q}{\gamma mc}\vec{v}_0 \times \hat{e}_z \left(\vec{x}_\perp \cdot \vec{\nabla}_\perp B\right) + \frac{q}{\gamma mc}\vec{v}_1 \times \vec{B}_0.$$

The first term on the right is $d\vec{v}_0/dt$, so

$$\frac{d\vec{v}_1}{dt} = \frac{q}{\gamma mc}\left(\vec{v}_1 \times \vec{B}_0 + \vec{v}_0 \times \hat{e}_z \left(\vec{x}_\perp \cdot \vec{\nabla}_\perp B\right)\right).$$

The unperturbed $\vec{x}_\perp$ is circular motion with radius a, with $\vec{v}_0 \times \hat{e}_z = -\omega_0 \vec{x}_\perp$. The average $\langle \vec{v}_0 \times \hat{e}_z(\vec{x}_\perp \cdot \vec{\nabla}_\perp B)\rangle = -\omega_0 \langle \vec{x}_\perp(\vec{x}_\perp \cdot \vec{\nabla}_\perp B)\rangle = -\frac{1}{2}\omega_0 a^2 \vec{\nabla}_\perp B$. We can find a constant drift velocity $\langle \vec{v}_1 \rangle$ on top of the oscillatory motion if $\langle d\vec{v}_1/dt\rangle = 0$. Thus

$$\langle \vec{v}_1 \rangle \times \vec{B}_0 = \frac{1}{2}\omega_0 a^2 \vec{\nabla}_\perp B,$$

or

$$\langle \vec{v}_1 \rangle = \frac{1}{B_0^2}\vec{B}_0 \times \left(\langle \vec{v}_1 \rangle \times \vec{B}_0\right) = \frac{\omega_0 a^2}{2B^2}\vec{B} \times \nabla_\perp B.$$

1.4 Bending Magnetic Fields

Now consider magnetic field lines which are gradually curving. At any point, we can consider the field line as having a center of curvature. Taking that as the origin of a polar coordinate system, the particle is at the point (ρ, ϕ, z) with the field pointing in the $\hat{e}_\phi$ direction. Once again the speed and γ are constant, so

$$\begin{aligned}\frac{d\vec{v}}{dt} &= \frac{1}{m\gamma}\frac{d\vec{p}}{dt} = \frac{q}{m\gamma}\vec{v} \times \vec{B} = \frac{qB}{m\gamma}\left(v_\rho \hat{e}_z - v_z \hat{e}_\rho\right) = \frac{qB}{m\gamma}\left(\dot{\rho}\hat{e}_z - \dot{z}\hat{e}_\rho\right) \\ &= \frac{d}{dt}\left(\dot{\rho}\hat{e}_\rho + \rho\dot{\phi}\hat{e}_\phi + \dot{z}\hat{e}_z\right) = \ddot{\rho}e_\rho + 2\dot{\rho}\dot{\phi}\hat{e}_\phi + \rho\ddot{\phi}\hat{e}_\phi - \rho\dot{\phi}^2\hat{e}_\rho + \ddot{z}\hat{e}_z,\end{aligned}$$

where we have used $d\hat{e}_\rho = \dot{\phi}e_\phi$, $d\hat{e}_\phi = -\dot{\phi}e_\rho$, $d\hat{e}_z = 0$. The ϕ component gives $2\dot{\rho}\dot{\phi} + \rho\ddot{\phi} = 0$ or $\rho^2\dot{\phi} = Rv_\parallel$, a constant. The other two components satisfy

$$\ddot{\rho} - \rho\dot{\phi}^2 = -\frac{qB}{m\gamma}\dot{z}, \qquad \ddot{z} = \frac{qB}{m\gamma}\dot{\rho}.$$

If we have motion[4] with $\rho \approx R$, averaging so we can ignore $\ddot{\rho}$, we have from the first equation that

$$\langle \dot{z} \rangle \approx \frac{m\gamma v_{\|}^2}{qBR}.$$

So we have a drift, again in a direction perpendicular to the center of curvature and to the direction of the field.

The last approximation we wish to consider uses the adiabatic invariance of the action. The action involved is $\oint \vec{P}_\perp \cdot d\vec{r}_\perp$ for the motion in the plane perpendicular to the field lines. But before we can discuss this, we need to know the **canonical** momentum $\vec{P}$ conjugate to $\vec{r}$, which **is not** the ordinary momentum $\vec{p} = m\gamma\vec{u}$. To find the **canonical momentum** we need to discuss the Lagrangian.

[4]Jackson assumes $B_\phi = B_0/\rho$, which is suitably curlless, and enables him to solve, with $x = \rho - R \ll R$, for x and show it is bounded.