Physics 504, Lecture 13
March 8, 2010

1 More on Lorentz Transformations

We saw last time that a (proper) finite Lorentz transformation $A^\mu{}_\nu$ can be thought of as being built up from an infinite number of infinitesimal transformations,

$$A^\mu{}_\nu = \lim_{N\to\infty}\left[\left(\delta^{\cdot}_{\cdot} + \frac{\omega}{N}L^{\cdot}_{\cdot}\right)^N\right]^\mu{}_\nu = e^{\omega L^{\cdot}_{\cdot}},$$

where the matrix $L^\mu{}_\nu$ needs to satisfy $L_{\mu\nu} = -L_{\nu\mu}$, and have real matrix elements. Thus the vector space of such infinitesimal generators can be described by a basis of such matrices, $\mathcal{L}_{\alpha\beta}$, so

$$L^\mu{}_\nu = \sum_{\alpha\beta} c^{\alpha\beta}\left(\mathcal{L}_{\alpha\beta}\right)^\mu{}_\nu\,,$$

where $c^{\alpha\beta}$ is antisymmetric, $c^{\alpha\beta} = -c^{\beta\alpha}$, and[1]

$$\left(\mathcal{L}_{\alpha\beta}\right)^{\mu\nu} = \delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta.$$

Note carefully where some indices have moved from up to down. The six independent $\mathcal{L}_{\alpha\beta}$ is *each* a 4×4 matrix, not the $\alpha\beta$ matrix element of one. The $\mathcal{L}_{\alpha\beta}$ with one index 0 generate Lorentz boosts, while those with two spatial indices generate rotations. Thus we may choose to look at these as two sets of three-vectors,

$$\left(K_i\right)^{\cdot}_{\cdot} = \left(\mathcal{L}_0{}^i\right)^{\cdot}_{\cdot}\,, \qquad \left(S_i\right)^{\cdot}_{\cdot} = -\frac{1}{2}\epsilon_{ijk}\left(\mathcal{L}_j{}^k\right)^{\cdot}_{\cdot}\,.$$

In problem 11.10 and, for part b, the similar statement for S, you can find the forms for an arbitrary Lorentz boost or an arbitrary rotation (but not an arbitrary mixture of them!).

The generators S_i of rotations should be familiar (with an i added) from quantum mechanics, where like all angular momentum operators they satisfy

[1] The L of Jackson 11.90 is $L^\mu{}_\nu = \left(L_\alpha{}^\beta\right)^\mu{}_\nu$.

commutation relations. Here, due to the extra i, we have $[S_i, S_j] = \epsilon_{ijk}S_k$. More generally,

$$\begin{aligned}
\left[\mathcal{L}_{\alpha\beta}, \mathcal{L}_{\gamma\zeta}\right]^\mu{}_\nu &= \Big(\Big\{\Big(\delta^\mu_\alpha\eta_{\beta\rho}\delta^\rho_\gamma\eta_{\zeta\nu} - (\alpha\leftrightarrow\beta)\Big) - (\gamma\leftrightarrow\zeta)\Big\} \\
&\qquad -(\alpha\leftrightarrow\gamma \text{ and } \beta\leftrightarrow\zeta)\Big) \\
&= \delta^\mu_\alpha\eta_{\beta\gamma}\eta_{\zeta\nu} - \delta^\mu_\beta\eta_{\alpha\gamma}\eta_{\zeta\nu} - \delta^\mu_\alpha\eta_{\beta\zeta}\eta_{\gamma\nu} + \delta^\mu_\beta\eta_{\alpha\zeta}\eta_{\gamma\nu} \\
&\qquad -\delta^\mu_\gamma\eta_{\zeta\alpha}\eta_{\beta\nu} + \delta^\mu_\zeta\eta_{\gamma\alpha}\eta_{\beta\nu} + \delta^\mu_\gamma\eta_{\zeta\beta}\eta_{\alpha\nu} \\
&= \eta_{\beta\gamma}\left(\delta^\mu_\alpha\eta_{\zeta\nu} - \delta^\mu_\zeta\eta_{\alpha\nu}\right) - \eta_{\beta\zeta}\left(\delta^\mu_\alpha\eta_{\gamma\nu} - \delta^\mu_\gamma\eta_{\alpha\nu}\right) \\
&\qquad -\eta_{\alpha\gamma}\left(\delta^\mu_\beta\eta_{\zeta\nu} - \delta^\mu_\zeta\eta_{\beta\nu}\right) + \eta_{\alpha\zeta}\left(\delta^\mu_\beta\eta_{\gamma\nu} - \delta^\mu_\gamma\eta_{\beta\nu}\right) \\
&= \eta_{\beta\gamma}\left(\mathcal{L}_{\alpha\zeta}\right)^\mu{}_\nu - \eta_{\beta\zeta}\left(\mathcal{L}_{\alpha\gamma}\right)^\mu{}_\nu - \eta_{\alpha\gamma}\left(\mathcal{L}_{\beta\zeta}\right)^\mu{}_\nu + \eta_{\alpha\zeta}\left(\mathcal{L}_{\beta\gamma}\right)^\mu{}_\nu\,.
\end{aligned}$$

There are several observations to be made about this expression. First, we see that the commutator of two generators is a linear superposition of generators. This is a feature of a Lie Algebra, which constitutes the generators of a continuous, or Lie, group. Of course the group of symmetry transformations which leave the invariant length invariant must be a group. Second, if we consider a rotation $\mathcal{L}_{ij} = \epsilon_{ijk}S_k$ commuted with a Lorentz boost $\mathcal{L}_{0\ell} = -K_\ell$, we see that

$$[S_k, K_\ell] = -\frac{1}{2}\epsilon_{ijk}\left[\mathcal{L}_{ij}, \mathcal{L}_{0\ell}\right] = \frac{1}{2}\epsilon_{ijk}\left\{\eta_{j\ell}\mathcal{L}_{i0} - \eta_{i\ell}\mathcal{L}_{j0}\right\} = \epsilon_{k\ell i}K_i,$$

so the vector of Lorentz boosts rotates as any vector should. If we consider the commutator of two Lorentz transformations,

$$[K_i, K_j] = [\mathcal{L}_{0i}, \mathcal{L}_{0j}] = -\mathcal{L}_{ij} = -\epsilon_{ijk}S_k,$$

so the commutator of two Lorentz boosts is a rotation!

Now we need to discuss derivatives. In three dimensions we have the gradient operator $\vec\nabla$. In four dimensions we use a different notation,

$$\partial_\mu := \frac{\partial}{\partial x^\mu}.$$

That the derivative operator should be considered a covariant (rather than contravariant) vector is clear from the chain rule:

$$\partial'_\mu = \frac{\partial}{\partial x'^{\,\mu}} = \frac{\partial x^\nu}{\partial x'^{\,\mu}}\frac{\partial}{\partial x^\nu} = A_\mu{}^\nu\partial_\nu,$$

where I have used $A_\mu{}^\nu = \partial x^\nu/\partial x'^{\,\mu}$ derived last time.

1.1 Application to Electromagnetism

So we have learned that physical properties can be scalars (invariant under Lorentz transformations) or 4-vectors, an appropriate combination of a 3-vector and a 3-scalar, or more generally they can be tensors with several co- or contra-variant indices[2] We have already found that the momentum and energy of a particle are combined into the 4-vector $p^\alpha = (E/c, \vec{p})$, but what about the 3-vectors $\vec{E}$ and $\vec{B}$? From the Lorentz force law[3] for a particle of charge q,

$$\vec{F} = \frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

while the rate of change of the kinetic energy of the particle is the power provided by the electric field,

$$\frac{dE}{dt} = q\vec{E}\cdot\vec{v}.$$

Thus we have an expression for the contravariant proper-time derivative of the 4-momentum, which can be expressed in terms of the 4-velocity $U^\alpha = (c\gamma, \vec{v}\gamma)$ as

$$\frac{dp^\alpha}{d\tau} = \frac{dt}{d\tau}\frac{dp^\alpha}{dt} = q\frac{U^0}{c}\left(\vec{E}\cdot\vec{v}, \vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) = \frac{q}{c}\left(\vec{E}\cdot\vec{U}, U^0\vec{E} + \vec{U}\times\vec{B}\right).$$

[2] In quantum mechanics the wave function can also be a spinor, such as the wave function for an electron. In classical mechanics these do not arise, and we will not consider them. This does not prevent us from considering the spin σ of an electron, which is an operator on a spinor but is itself a 3-vector (or more properly, $\psi^\dagger\vec{\sigma}\psi$ is a 3-vector).

[3] Jackson has changed notation at this point — from now on we use Gaussian Units. The microscopic form of Maxwell's equations are now

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t},$$

the Lorentz force is

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right),$$

Coulomb's law is $\vec{E} = q\vec{r}/r^3$, the fields are related to the vector potential by

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla}\times\vec{A}, \quad \text{and} \quad \vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0$$

is now the Lorenz gauge condition.

We see that $\vec{E}$ and $\vec{B}$ cannot transform independently, for a particle at rest ignores any $\vec{B}$ while one in motion does not. The left hand side of this equation is a contravariant 4-vector which depends linearly on the velocity 4-vector, but is not proportional to it. So this calls out for a tensor quantity $F^\alpha{}_\beta$ in terms of which

$$\frac{dp^\alpha}{d\tau} = \frac{q}{c}F^\alpha{}_\beta U^\beta.$$

Matching terms we see

$$F^0{}_0 = 0, \quad F^0{}_i = E_i, \quad F^i{}_0 = E_i, \quad F^i{}_j = \epsilon_{ijk}B_k.$$

If we raise the second index or lower the first, we get

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \qquad F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}.$$

Note that F, which is called the field-strength tensor, is antisymmetric. Those who recall differential forms will be tempted to consider a 2-form $\mathbf{F} = \frac{1}{2}F_{\alpha\beta}dx^\alpha dx^\beta$ and ask what the exterior derivative is. It will be a three form associated with the vector

$$\frac{1}{12}\epsilon^{\alpha\beta\gamma\zeta}\partial_\beta F_{\gamma\zeta},$$

where $\epsilon^{\alpha\beta\gamma\zeta}$ is the totally antisymmetric Levi-Civita symbol for which[4] $\epsilon^{0123} = 1$. The zeroth component of 12 times this is

$$\epsilon^{ijk}\partial_i F_{jk} = \epsilon^{ijk}\partial_i(-1)\epsilon_{jk\ell}B_\ell = -2\vec{\nabla}\cdot\vec{B},$$

which vanishes according to one of Maxwell's laws. The i'th spatial component is

$$\begin{aligned}\epsilon^{i\beta\gamma\zeta}\partial_\beta F_{\gamma\zeta} &= \epsilon^{i0jk}\partial_0 F_{jk} + 2\epsilon^{ijk0}\partial_j F_{k0} = -\frac{1}{c}\epsilon_{ijk}\left(-\epsilon_{jk\ell}\frac{\partial B_\ell}{\partial t}\right) - 2\epsilon_{ijk}\partial_j(-E_k) \\ &= 2\left(\vec{\nabla}\times\vec{E}\right)_i + \frac{2}{c}\frac{\partial B_i}{\partial t},\end{aligned}$$

[4] In flat space. In general relativity $\epsilon^{0123} = 1/\sqrt{|\det(g_{\mu\nu})|}$.

which also vanishes, by another of Maxwell's laws. Thus $d\mathbf{F} = 0$ or

$$\frac{1}{2}\epsilon^{\alpha\beta\gamma\zeta}\partial_\beta F_{\gamma\zeta} = 0 \tag{1}$$

constitute the sourceless half of Maxwell theory. We could also express this in terms of the dual to the field-strength tensor,

$$\mathcal{F}^{\alpha\beta} := \frac{1}{2}\epsilon^{\alpha\beta\gamma\zeta}F_{\gamma\zeta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}.$$

This dual tensor can be viewed as the result of a duality that changes $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$. In terms of $\mathcal{F}$, (1) is

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0.$$

But Maxwell's equations are not invariant under this duality except in the absence of sources, for there are no magnetic monopoles, at least as far as we currently know. What is the equivalent of $d\mathcal{F}$ or Eq. 1 with $F \to \mathcal{F}$?

$$\epsilon^{\alpha\beta\gamma\zeta}\partial_\beta\mathcal{F}_{\gamma\zeta} = \epsilon^{\alpha\beta\gamma\zeta}\partial_\beta\epsilon_{\gamma\zeta\rho\sigma}\frac{1}{2}F^{\rho\sigma} = 3\left(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho\right)\partial_\beta F^{\rho\sigma} = 6\partial_\beta F^{\alpha\beta}$$

Well, the $\alpha = 0$ component of $\partial_\beta F^{\alpha\beta}$ is

$$\partial_j F^{0j} = -\vec{\nabla}\cdot\vec{E} = -4\pi\rho,$$

and the spatial components are

$$\partial_\beta F^{i\beta} = \frac{\partial F^{i0}}{c\partial t} + \partial_j(-\epsilon_{ijk}B_k) = \left(-\vec{\nabla}\times\vec{B} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)_i = -\frac{4\pi}{c}\vec{J}_i.$$

We see that we need to combine $\vec{J}$ and ρ into

$$j^\alpha = \left(c\rho, \vec{J}\right)$$

and we now have

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}j^\nu.$$

Notice this has a familiar immediate consequence:

$$\partial_\nu\left(\partial_\mu F^{\mu\nu}\right) = \frac{4\pi}{c}\partial_\nu j^\nu = 0$$

where the vanishing comes because $\partial_\nu\partial_\mu$ is symmetric under $\mu \leftrightarrow \nu$ while $F^{\mu\nu}$ is antisymmetric. Thus we see that

$$\partial_\nu j^\nu = 0 = \frac{\partial c\rho}{c\partial t} + \vec{\nabla}\cdot\vec{J},$$

the equation of continuity follows from Maxwell's equations.

Notice that we have the charge density ρ transforming like the zeroth component of a 4-vector. Is this right? Charge is invariant, so the charge in a given infinitesimal volume, $dq = \rho d^3x$ should be invariant, but the volume suffers a Fitzgerald contraction under Lorentz transformation. Indeed, the four-dimensional volume element $d^4x = dx^0 d^3x$ is invariant, because

$$d^4x' = \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) d^4x = \det\left(A^\mu{}_\nu\right) d^4x.$$

Taking the determinant of the condition for $A^\cdot{}_\cdot$ to be a Lorentz transformation,

$$\eta_{\alpha\beta}A^\alpha{}_\mu A^\beta{}_\nu = \eta_{\mu\nu} \tag{2}$$

we have $\det\eta_{\cdot\cdot}(\det A^\cdot{}_\cdot)^2 = \det\eta_{\cdot\cdot}$ or $\det A^\cdot{}_\cdot = \pm 1$. This brings up an issue we have neglected. Is any matrix satisfying (2) a Lorentz transformation? If $\mathcal{O}'$'s reference frame was originally boosted from $\mathcal{O}$'s by firing the rocket engines, the velocity relative to $\mathcal{O}$ and the Lorentz transformation should evolve continuously. As it starts with $A^\mu{}_\nu = \delta^\mu_\nu$ which has determinant 1, and a matrix with continuously varying matrix elements has its determinant varying continuously, the determinant cannot jump to -1 and must be 1. So d^4x is invariant and ρ transforms the same way dx^0 and x^0 do.

There is another constraint on A if this continuous connection is imposed. Taking the 00'th matrix element of (2), we have $\eta_{\mu\nu}A^\mu{}_0 A^\nu{}_0 = (A^0{}_0)^2 - (A^i{}_0)^2 = 1$, so $|A^0{}_0| \geq 1$. Again, starting at 1, it cannot vary continuously to get to a negative number.

We will call any matrix satisfying (2) a Lorentz transformation, but restrict our attention to those with determinant $+1$ (proper Lorentz transformations) and with $A^0{}_0 \geq 1$ (orthochronous Lorentz transformations). Note the latter is the condition that time runs in the same direction for both observers. The parity operation $\vec{x} \to -\vec{x}$, t unchanged, is an improper orthochronous Lorentz transformation, while time reversal together with parity, $\vec{x} \to -\vec{x}$, is non-orthochronous but proper. Physics, so far, is invariant under proper orthochronous Lorentz transformations, as Einstein wanted, but not the others (see Wu and Yang, Fitch and Cronin).

Getting back to the four dimensional form of Maxwell's equations, form-friendly observers having noted that $d\mathbf{F} = 0$ will be encouraged to ask what 1-form $\mathbf{A}$ satisfies $\mathbf{F} = d\mathbf{A}$, which, for the form-unfriendly among you, says

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3}$$

Clearly we are to suspect the 4-vector

$$A^\mu = \left(\Phi, \vec{A}\right),$$

where Φ is the electrostatic potential and $\vec{A}$ the usual vector potential. Indeed the $0j$ component of (3) says[5]

$$E_j = \frac{1}{c}\frac{\partial A_j}{\partial t} - \partial_j \Phi = -\left(\vec{\nabla}\Phi + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right)_j,$$

and the ij component gives[5]

$$-\epsilon_{ijk} B_k = \partial_i A_j - \partial_j A_i = -\epsilon_{ijk}\left(\vec{\nabla} \times \vec{A}\right)_k$$

The Lorenz gauge condition is

$$0 = \vec{\nabla} \cdot \vec{A} + \frac{1}{c}\frac{\partial A^0}{\partial t} = \partial_\mu A^\mu.$$

Finally, the operator for the wave equation in empty space is

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = -\partial_\mu \partial^\mu =: -\Box.$$

[5] Note A_μ for $\mu = j$ is $A_j = -A^j = -(\vec{A})_j$. I apologize for this confusing notation, one reason for preferring the opposite choice of sign for η from the one Jackson chooses.