It is easy to evaluate the 27 coefficients $\epsilon_{kij}$, because the cross product of two orthogonal unit vectors is a unit vector orthogonal to both of them. Thus $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$, so $\epsilon_{312} = 1$ and $\epsilon_{k12} = 0$ if $k = 1$ or 2. Applying the same argument to $\hat{e}_2 \times \hat{e}_3$ and $\hat{e}_1 \times \hat{e}_3$, and using the antisymmetry of the cross product, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$, we see that

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1; \quad \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1,$$

and $\epsilon_{ijk} = 0$ for all other values of the indices, i.e. $\epsilon_{ijk} = 0$ whenever any two of the indices are equal. Note that $\epsilon$ changes sign not only when the last two indices are interchanged (a consequence of the antisymmetry of the cross product), but whenever any two of its indices are interchanged. Thus $\epsilon_{ijk}$ is zero unless $(1, 2, 3) \to (i, j, k)$ is a permutation, and is equal to the sign of the permutation if it exists.

Now that we have an expression for $\hat{e}_i \times \hat{e}_j$, we can evaluate

$$\vec{A} \times \vec{B} = \sum_i \sum_j \epsilon_{ijk} A_i B_j \hat{e}_k.$$

Much of the usefulness of expressing cross products in terms of $\epsilon$'s comes from the identity

$$\sum_k \epsilon_{kij} \epsilon_{k\ell m} = \delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell},$$

which can be shown as follows. To get a contribution to the sum, $k$ must be different from the unequal indices $i$ and $j$, and also different from $\ell$ and $m$. Thus we get 0 unless the pair $(i, j)$ and the pair $(\ell, m)$ are the same pair of different indices. There are only two ways that can happen, as given by the two terms, and we only need to verify the coefficients. If $i = \ell$ and $j = m$, the two $\epsilon$'s are equal and the square is 1, so the first term has the proper coefficient of 1. The second term differs by one transposition of two indices on one epsilon, so it must have the opposite sign.

We now turn to some applications. Let us first evaluate

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \sum_{ij} \sum_k \epsilon_{ijk} B_j C_k = \sum_{ijk} \epsilon_{ijk} A_i B_j C_k.$$

Note that $\vec{A} \cdot (\vec{B} \times \vec{C})$ is, up to sign, the volume of the parallelopiped formed by the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$. From the fact that the $\epsilon$ changes sign under
transpositions of any two indices, we see that the same is true for transposing
the vectors, so that
\[ \vec{A} \cdot (\vec{B} \times \vec{C}) = -\vec{A} \cdot (\vec{C} \times \vec{B}) = -\vec{B} \cdot (\vec{C} \times \vec{A}) = -\vec{B} \cdot (\vec{A} \times \vec{C}) = -\vec{C} \cdot (\vec{A} \times \vec{B}) = -\vec{C} \cdot (\vec{B} \times \vec{A}). \]

Now consider \( \vec{V} = \vec{A} \times (\vec{B} \times \vec{C}) \). Using our formulas,
\[
\vec{V} = \sum_{ijk} \epsilon_{kij} \hat{e}_k A_i \sum_{lm} \epsilon_{jlm} B_l C_m.
\]
Notice that the sum on \( j \) involves only the two epsilons, and we can use
\[
\sum_j \epsilon_{kij} \epsilon_{jlm} = \sum_j \epsilon_{jki} \epsilon_{jlm} = \delta_{kl} \delta_{im} - \delta_{km} \delta_{il}.
\]
Thus
\[
V_k = \sum_{ilm} (\sum_j \epsilon_{kij} \epsilon_{jlm}) A_i B_l C_m = \sum_{ilm} (\delta_{kl} \delta_{im} - \delta_{km} \delta_{il}) A_i B_l C_m
= \sum_{ilm} \delta_{kl} \delta_{im} A_i B_l C_m - \sum_{ilm} \delta_{km} \delta_{il} A_i B_l C_m
= \sum_i A_i B_k C_i - \sum_i A_i B_l C_k = \vec{A} \cdot \vec{C} B_k - \vec{A} \cdot \vec{B} C_k,
\]
so
\[ \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} \vec{A} \cdot \vec{C} - \vec{C} \vec{A} \cdot \vec{B}. \] (7)

This is sometimes known as the \textbf{bac-cab} formula.

Exercise: Using (5) for the manipulation of cross products, show
that
\[
(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = \vec{A} \cdot \vec{C} \vec{B} \cdot \vec{D} - \vec{A} \cdot \vec{D} \vec{B} \cdot \vec{C}.
\]

The determinant of a matrix can be defined using the \( \epsilon \) symbol. For a
3 \( \times \) 3 matrix \( A \),
\[
\det A = \sum_{ijk} \epsilon_{ijk} A_{i1} A_{2j} A_{3k} = \sum_{ijk} \epsilon_{ijk} A_{i1} A_{j2} A_{k3}.
\]
From the second definition, we see that the determinant is the volume of the
parallelopiped formed from the images under the linear map \( A \) of the three
unit vectors \( \hat{e}_i \), as
\[
(A \hat{e}_1) \cdot ((A \hat{e}_2) \times (A \hat{e}_3)) = \det A.
\]