Notes on Bessel Functions
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Bessel functions $J_m(x)$ of integral order $m$ may be defined by the generating function

$$g(x, t) := e^{x/2}(t - 1/t) = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

As the generating function is unchanged by $x \to -x, t \to 1/t$, we have $J_{-n}(x) = J_n(x)$, but it is also unchanged by $x \to -x, t \to -t$, so $J_{-n}(x) = (-1)^n J_n(x)$, so $J_{-n}(x) = (-1)^n J_n(x)$

Differentiating (1) with respect to $t$ gives

$$\frac{1}{2} \left( 1 + \frac{1}{t^2} \right) g(x, t) = \sum_{n=0}^{\infty} n J_n(x) t^{n-1} = \frac{x}{2} \sum_{n=0}^{\infty} J_n \left( t^n + t^{n+2} \right)$$

$$\implies J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x).$$  (2)

Differentiating (1) with respect to $x$ gives

$$\frac{1}{2} \left( t - \frac{1}{t} \right) g(x, t) = \sum_{n=0}^{\infty} J'_n(x) t^n = \sum_{n=0}^{\infty} J_n(x) \left( t^{n+1} - t^{n-1} \right)$$

$$\implies J_{n-1}(x) - J_{n+1}(x) = 2 J'_n(x).$$  (3)

As a special case, $J'_0(x) = -J_1(x)$. (2) $\iff$ (3) gives

$$J_{n\pm 1}(x) = \frac{n}{x} J_n(x) \mp J'_n(x).$$  (4)

Manipulating Eqs. (4), even if $m$ is not an integer, gives the Bessel equation

$$x^2 \frac{d^2}{dx^2} J_v(x) + x \frac{d}{dx} J_v(x) + (x^2 - v^2) J_v(x) = 0$$

This can also be written

$$u \frac{d}{d u} u \frac{d}{d u} J_v(\alpha u) = (\nu^2 - \alpha^2 u^2) J_v(\alpha u)$$

Orthogonality:

Then

$$J_v(\alpha u) \frac{d}{d u} u \frac{d}{d u} J_v(\beta u) - J_v(\beta u) \frac{d}{d u} u \frac{d}{d u} J_v(\alpha u) = (\alpha^2 - \beta^2) u J_v(\alpha u) J_v(\beta u)$$

Integrate from 0 to 1:

$$\int_0^1 du \ u J_v(\alpha u) J_v(\beta u) = \int_0^1 u J_v(\alpha u) \frac{d}{d u} u \frac{d}{d u} J_v(\beta u) - (\alpha \leftrightarrow \beta)$$

$$= u J_v(\alpha u) \frac{d}{d u} J_v(\beta u) \bigg|_0^1 - \int_0^1 u \left( \frac{d}{d u} J_v(\alpha u) \right) \frac{d}{d u} J_v(\beta u) \bigg|_0^1 - (\alpha \leftrightarrow \beta)$$

$$= u J_v(\alpha u) \frac{d}{d u} J_v(\beta u) \bigg|_0^1 - u J_v(\beta u) \frac{d}{d u} J_v(\alpha u) \bigg|_0^1.$$

(6)

For $\nu \geq 0$ we may assume $J_v(0)$ is finite, so the lower endpoint gives zero. If $\alpha$ and $\beta$ are both zeros of $J_v$ or both zeros of $J'_v$, the upper endpoint also vanishes, so

$$\int_0^1 du \ u J_v(x_{\nu m} u) J_v(x_{\nu m} u) = 0$$

$$\int_0^1 du \ u J_v(x'_{\nu m} u) J_v(x'_{\nu m} u) = 0$$

for $m \neq n, \nu \geq 0$

If we first differentiate (6) with respect to $\alpha$, and then set $\beta = \alpha$, we get

$$2\alpha \int_0^1 du \ u J'_v(\alpha u) = u^2 J'_v(\alpha u) \frac{d}{d u} J_v(\alpha u) \bigg|_0^1 - u J_v(\alpha u) \frac{d}{d u} \left( \frac{u d}{d u} J_v(\alpha u) \right) \bigg|_0^1,$$

where we need to be careful that $dJ_v(\alpha u)/d\alpha = uJ'_v(\alpha u) = \frac{u d}{d u} J_v(\alpha u)$. The lower endpoint vanishes. Using the Bessel equation gives

$$2\alpha \int_0^1 du \ u J'_v(\alpha u) = \alpha J'_v(\alpha u) + J_v(\alpha) \frac{\alpha^2 - \nu^2}{\alpha} J_v(\alpha).$$

Substitute $J'_v(\alpha) \to \nu \alpha J_v(\alpha) - J_{v+1}(\alpha)$ to get

$$\int_0^1 du \ u J'_v(\alpha u) = \frac{1}{2} \left( \nu J_v(\alpha) - J_{v+1}(\alpha) \right)^2 + \frac{\alpha^2 - \nu^2}{2\alpha^2} J'_v(\alpha)$$

$$= \frac{1}{2} J_{v+1}^2(\alpha) - \nu \alpha J_v(\alpha) J_{v+1}(\alpha) + \frac{1}{2} J_v^2(\alpha).$$

(7)
If we set $\alpha$ to a zero of $J_\nu$, the last two terms vanish and

$$
\int_0^1 du \, u J_\nu^2(x_\nu u) = \frac{1}{2} J_\nu^2(x_\nu u). \quad (8)
$$

If we set $\alpha$ to $x'_\nu$, a zero of $J'_\nu = \frac{\nu}{x_\nu} J_\nu(x'_\nu) - J_{\nu+1}(x'_\nu)$, so $J_{\nu+1}(x'_\nu) = \frac{\nu}{x_\nu} J_\nu(x'_\nu)$, and (7) becomes

$$
\int_0^1 du \, u J'_\nu(x'_\nu u) = \frac{1}{2} \left[ 1 - \left( \frac{\nu}{x'_\nu} \right)^2 \right] J'_\nu(x'_\nu) \quad (9)
$$