Lecture 13  March 7, 2011

Last time we discussed a small scatterer at origin. Interesting effects come from many small scatterers occupying a region of size $d$ large compared to $\lambda$. The scatterer $j$ at position $\vec{x}_j$ has an $E_{\text{inc}}$ with an extra factor of $e^{i\vec{k}\cdot\vec{x}_j}$, and in the scattered wave, $\vec{r}$ needs to be replaced by $\vec{r} - \vec{x}_j$. Assuming we are observing from far away, $|\vec{r}| \gg d$, the variations of the $r$ in the denominator or the $i\vec{r}$'s are not important, but the effect in the oscillating exponential is, and we should approximate

$$e^{i\vec{k}\cdot\vec{r}} \approx e^{i\vec{k}\cdot\vec{r} - i\vec{k}\cdot\vec{x}_j}$$

So the amplitude for the scattered wave due to $j$ has an extra factor of

$$e^{i\vec{k}\cdot\vec{r} - i\vec{k}\cdot\vec{x}_j} = e^{i\vec{q}\cdot\vec{r}}, \quad \text{with} \quad \vec{q} = k(\vec{n}_j - \vec{r})$$

The amplitudes for all the scatterers need to be added before squaring to find the flux, so we have

$$\frac{de}{dt} = \frac{k^4}{(4\pi\varepsilon_0\mu_0)^2} \sum_{j} \left( |e^{i\vec{q}\cdot\vec{n}_j}|^2 + (i\vec{q} \cdot \vec{E}_j) \frac{\vec{E}_j}{|\vec{E}_j|^2} \right)^2$$

If all the scatterers react the same way, $\vec{p}_j$ and $m_j$ can be factored out of the sum, and we appear to have a single scatterer with a structure factor

$$\mathcal{F}(\vec{q}) = \left| \sum_j e^{i\vec{q}\cdot\vec{r}_j} \right|^2 = \sum_j \sum_j e^{i\vec{q}\cdot(\vec{r}_j - \vec{r}_j')}$$

The nature of $\mathcal{F}(\vec{q})$ depends on how the scatterers are distributed.

Structure Factor

- Large number of randomly positioned scatterers: phases random — superposition incoherent.
  - Only the terms with $i = j$ contribute, $\mathcal{F}(\vec{q}) = N$, except for $\vec{q} = 0$. Coherent scattering $\approx N^2$, so incoherent scattering is very faint.
- Crystaline structure: with a regular array we can get even less scattering.
  - Consider a one dimensional array of $N$ scatterers each displaced by $\vec{a}$ from the previous.

$$\mathcal{F}(\vec{q}) = \left| \sum_{j=0}^{N-1} e^{i\vec{q}\cdot\vec{a}} \right|^2 = \left| 1 - e^{iN\vec{q}\cdot\vec{a}} \right|^2 = N^2 \sin^2(N\vec{q}\cdot\vec{a}/2) / (N \sin(q\cdot\vec{a}/2))^2$$

For lattice spacings $a \ll \lambda$ but total extent $Na \gg \lambda$, the fraction is $(\sin x/x)^2$ for $x = N\vec{q}\cdot\vec{a}/2$. $x \gg 1$ and $(\sin x/x)^2 \ll 1$ unless $\vec{q}\cdot\vec{a}$ is comparable or smaller than $1/N$.

Applying Maxwell

Maxwell in medium but without sources applies:

As $\nabla \cdot \vec{D} = 0$,

$$\nabla^2 \vec{D} = -\nabla \times (\nabla \times \vec{D}) = -\nabla \times (\nabla \times (\vec{B} - \vec{\epsilon} \vec{E}))$$

last term: $\vec{\epsilon} \nabla \times \frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t} \left( \vec{B} - \vec{\mu} \vec{H} \right) + \vec{\mu} (\vec{H} \times \vec{H} \times \vec{B})$

So altogether,

$$\nabla^2 \vec{D} - \epsilon \nabla^2 \vec{E} = -\nabla \times (\nabla \times (\vec{B} - \vec{\epsilon} \vec{E}) + \mu \nabla \times (\vec{B} - \vec{\mu} \vec{H}))$$

This equation is exact. Good approximations: $\delta \epsilon, \delta \mu$ small, treat to first order, as sources. Can treat full field $\vec{D}$ as harmonic, $\propto e^{-i\omega t}$ so $\vec{D}$ satisfies inhomogeneous Helmholtz equation with $k^2 := \vec{p}\vec{\epsilon} \vec{a}^2$, and all fields perturbations on an incident plane wave

$$\vec{B}_{\text{inc}}(\vec{x}) = \vec{B}_i e^{i\delta\vec{\epsilon} \cdot \vec{x}}$$

the fields in the source term, to first order in the variations, will be

$$\vec{B} - \vec{\epsilon} \vec{E} = \frac{\delta \epsilon(\vec{x})}{\epsilon} \vec{B}_{\text{inc}}(\vec{x})$$

$$\vec{B} - \vec{\mu} \vec{H} = \frac{\delta \mu(\vec{x})}{\mu} \vec{B}_{\text{inc}}(\vec{x})$$
the correction will then be the scattered wave given by the Green's function

$$\tilde{D} - \tilde{D}_{\text{inc}} = \frac{1}{4\pi} \int d^3x' e^{i(k|\vec{x} - \vec{x}'|)} \left\{ \frac{1}{\epsilon} \frac{\vec{\nabla} \times \vec{\nabla} - k^2}{\vec{\nabla} \times \vec{\nabla} + k^2} \times \left( \delta(t \prime) \tilde{D}_{\text{inc}}(\vec{x}' \prime) \right) \right\} \cdot \vec{A}(\vec{x} \prime) \times \tilde{A}(\vec{x} \prime)$$

Integration by parts: Note \( \int_V \frac{\vec{\nabla} \times \vec{A} = \int_{\partial V} \vec{A} \times \vec{n} \rightarrow 0 \) if \( \vec{A} \) vanishes sufficiently at infinity, and therefore

$$\int_V d^3x' \vec{\nabla} \times \vec{A}(\vec{x} \prime) \times \tilde{A}(\vec{x} \prime) \sim - \int_V d^3x' \left( \vec{\nabla} \times f(\vec{x}') \right) \cdot \vec{\nabla} \times \tilde{A}(\vec{x} \prime).$$

For the \( \tilde{D}_{\text{inc}} \) term, we also need

$$\int_V d^3x' f(\vec{x}') \vec{\nabla} \times \vec{\nabla} \times \tilde{A}(\vec{x} \prime) = \int_V d^3x' f(\vec{x}') \left( \vec{\nabla} \times \tilde{A}(\vec{x} \prime) \right) \times \vec{A}(\vec{x} \prime) - k^2 \int_V d^3x' \tilde{A}(\vec{x} \prime) \vec{\nabla} \times f(\vec{x}') \cdot \tilde{A}(\vec{x} \prime) \times \vec{A}(\vec{x} \prime)$$

Again \( f(\vec{x}') = e^{i \vec{k}(\vec{x}' - \vec{x}_0)/|\vec{x}' - \vec{x}_0|} \) is the Green's function for \( \nabla^2 + k^2 \); so for the second term, outside the region of scattering (where we can ignore the \( \delta(\vec{x} - \vec{x}') \) term) we have \( k^2 \int_V d^3x' \tilde{A}(\vec{x} \prime) e^{i \vec{k}(\vec{x}' - \vec{x}_0)/|\vec{x}' - \vec{x}_0|} \times \tilde{A}(\vec{x} \prime) \).

For large \( r \), we have

$$e^{i \vec{k}(\vec{x}' - \vec{x})} \approx e^{i \vec{k}_r \cdot \vec{x}} = \frac{1}{r},$$

$$\vec{\nabla} f(\vec{x}') \sim - \frac{i \vec{k}}{r^2} e^{i \vec{k}_r \cdot \vec{x}} \text{, and } \left( \vec{A} \cdot \vec{\nabla} \right) \left( \vec{\nabla} f(\vec{x}') \right) \sim - \frac{k^2}{r} \vec{A} e^{i \vec{k}_r \cdot \vec{x}} e^{i \vec{k}_r \cdot \vec{x}}.$$ 

So altogether

$$\tilde{D} = \tilde{D}_{\text{inc}} + \frac{e^{i \vec{k}_r \cdot \vec{x}}}{r} \tilde{A}_{\text{inc}},$$

where

$$\tilde{A}_{\text{inc}} = \frac{k^2}{4\pi} \int d^3x' e^{i \vec{k}_r \cdot \vec{x}} \left\{ \frac{\delta(\vec{x}')}{\epsilon} \left( \hat{r} \times \tilde{D}_{\text{inc}}(\vec{x}' \prime) \right) \times \hat{r} \right. \right.$$
The intensity of the beam \( I(z) = I(0)e^{-\alpha z} \) falls exponentially with distance with the attenuation coefficient \( \alpha \) due to the scattering. In a slice of width \( dz \), there are \( N dz \) scatters per unit area, each scattering an area \( \sigma \) of the beam, so there is a fractional loss of \( N \sigma dz \) in distance \( dz \), and

\[
\alpha = N \sigma \approx \frac{2k^4}{3\pi N} \left| \epsilon - 1 \right|^2.
\]

This is Rayleigh scattering. Note that it is a method of determining the number of molecules, so an approach which was used historically to determine Avagadro’s number.

In a fluid in equilibrium with a reservoir at constant pressure and temperature, the probability that a given piece of fluid occupies a volume \( V \) is \( \exp(-G(V)/k_BT) \), where \( G \) is the Gibbs free energy and \( k_B \) is Boltzmann’s constant.

In terms of the isothermal compressibility

\[
\beta_T = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_T = \left( \frac{\partial^2 G}{\partial V^2} \right)^{-1},
\]

the mean square deviation of \( \langle (\Delta V)^2 \rangle \) is \( k_BT \langle \Delta V \rangle \beta_T \), and

\[
\langle (\Delta N)^2 \rangle = k_BT \langle N^2/V \rangle \beta_T.
\]

\[\text{See Ref. p300}\]

Critical Opalescence

In the previous discussion we assumed no correlation in the positions of the scatterers. This is not a good approximation in denser fluids. A better approximation is to consider \( \epsilon \) to be the mean permittivity of the fluid but take into account density fluctuations. From the Clausius-Mossotti relation (14.70) we have

\[
\epsilon_r = \frac{3 + 2N_{\text{mol}}}{3 - N_{\text{mol}}} \Rightarrow \frac{d\epsilon_r}{dN} = \frac{9N_{\text{mol}}}{(3-N_{\text{mol}})^2} \epsilon_r - 1 \left( \epsilon_r \right)^2 \frac{2}{3N \epsilon_r},
\]

so the variation of \( \epsilon \) in a region of fluid with varying density is

\[
\frac{\delta \epsilon}{\epsilon_0} = \frac{(\epsilon_r - 1)(\epsilon_r + 2)}{3N} \delta N.
\]

How do we evaluate \( \delta N? \)

As for the blue sky, the attenuation coefficient is just \( \alpha = N \sigma \) and the angular integral is

\[
\int d\Omega |\epsilon^* \cdot \epsilon^*| = 8\pi/3,
\]

so

\[
\alpha = \frac{k^4}{6\pi^2 N} \left( \epsilon_r - 1 \right)^2 \frac{2}{3} N k_B T \beta_T
\]

\[
= \frac{\omega^2}{6\pi N c^4} \left( \epsilon_r - 1 \right)^2 \frac{2}{3} N k_B T \beta_T.
\]

The most important feature of this is that at the critical point the compressibility \( \beta_T \) blows up, so the fluid becomes opalescent.

I am going to skip the sections on diffraction. This has been or is covered in our optics courses.