Lecture 12 March 3, 2011

- We will first finish up the $\ell = 1$ term from the Green function. This is giving us magnetic dipole and electric quadripole contributions.
- We will briefly describe a more consistent way of doing the angular expansion, using "vector spherical harmonics". This will more cleanly separate magnetic multipoles and electric multipoles, and is consistent with what we did for the spherical cavity.
- We will skip most of the rest of chapter 9 and go on to scattering of the electromagnetic waves.

Vector Spherical Harmonics

Last time, used scalar Green function on vector source. This mixes spherical expansion with vectors in an awkward way,
For example, $\ell = 1$ mixed magnetic dipole and electric quadripole source contributions.

In doing spherical cavity, we expanded scalars, $\vec{r} \cdot \vec{E}$ and $\vec{r} \cdot \vec{H}$. Each satisfies Helmholtz away from sources, so for $r > d$ they are expanded in spherical bessels times spherical harmonics, as we learned in lecture 5:

$$\vec{r} \cdot \vec{H}^{(M)}_{\ell m} = \frac{\ell(\ell+1)}{k} g_\ell(kr) Y_{\ell m}(\theta, \phi), \qquad \vec{r} \cdot \vec{E}^{(M)} = 0$$

$$\text{or} \quad \vec{r} \cdot \vec{E}^{(E)}_{\ell m} = -Z_0 \frac{\ell(\ell+1)}{k} g_\ell(kr) Y_{\ell m}(\theta, \phi), \qquad \vec{r} \cdot \vec{H}^{(E)} = 0$$

for magnetic multipole modes (M) or electric multipole modes (E).

In either case g_ℓ satisfies the spherical Bessel equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} + k^2\right) g_\ell(kr) = 0$$

with solutions outside the source region proportional to $h_\ell^{(1)}(kr)$ for outgoing waves. We found the transverse components are given by

$$\vec{E}^{(M)}_{\ell m} = Z_0 g_\ell(kr)\vec{L}Y_{\ell m}, \qquad \vec{H}^{(M)}_{\ell m} = -\frac{i}{kZ_0}\vec{\nabla}\times\vec{E}^{(M)}_{\ell m}$$

or $$\vec{H}^{(E)}_{\ell m} = g_\ell(kr)\vec{L}Y_{\ell m}(\theta,\phi), \qquad \vec{E}^{(E)}_{\ell m} = i\frac{Z_0}{k}\vec{\nabla}\times\vec{H}^{(E)}_{\ell m}$$

where $\vec{L} = -i\vec{r}\times\vec{\nabla}$.

For $\ell \geq 1$, define: $$\vec{X}_{\ell m}(\theta,\phi) := \frac{1}{\sqrt{\ell(\ell+1)}}\vec{L}Y_{\ell m}(\theta,\phi).$$

These are called the *vector spherical harmonics*, and provide good basis functions for expanding our fields.

Orthogonality Properties

$$
\begin{aligned}
\int d\Omega \vec{X}^*_{\ell' m'} \cdot \vec{X}_{\ell m} &= \frac{1}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}} \\
&\quad \times \int d\Omega \Big[\frac{1}{2} \left(L_+^* Y^*_{\ell' m'} \right) \left(L_+ Y_{\ell m} \right) \\
&\qquad + \frac{1}{2} \left(L_-^* Y^*_{\ell' m'} \right) \left(L_- Y_{\ell m} \right) \\
&\qquad + \left(L_z^* Y^*_{\ell' m'} \right) \left(L_z Y_{\ell m} \right) \Big] \\
&= \int d\Omega \, \frac{Y^*_{\ell' m'} \left[\frac{1}{2} L_- L_+ + \frac{1}{2} L_+ L_- + L_z^2 \right] Y_{\ell m}}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}} \\
&= \frac{\sqrt{\ell(\ell+1)}}{\sqrt{\ell'(\ell'+1)}} \int d\Omega \, Y^*_{\ell' m'} Y_{\ell m} \\
&= \delta_{\ell \ell'} \delta_{m m'}
\end{aligned}
$$

where $\int d\Omega = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi$, and we have used

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Multipole Expansion

Scattering

$\vec{L}^2 = \frac{1}{2}L_-L_+ + \frac{1}{2}L_+L_- + L_z^2$, $\int d\Omega(\vec{L}\Phi)^*\Psi = \int d\Omega\Phi^*\vec{L}\Psi$, and $\int d\Omega Y^*_{\ell' m'}Y_{\ell m} = \delta_{\ell\ell'}\delta_{mm'}$.
So the vector spherical harmonics are an orthonormal set:

$$\int d\Omega \vec{X}^*_{\ell' m'} \cdot \vec{X}_{\ell m} = \delta_{\ell\ell'}\delta_{mm'}.$$

We also have

$$\begin{aligned}
&\int d\Omega \vec{X}^*_{\ell' m'} \cdot \left(\vec{r} \times \vec{X}_{\ell m}\right) \\
&\quad = \frac{1}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}} \int d\Omega \left(\vec{L}^* Y^*_{\ell' m'}\right) \cdot (\vec{r} \times \vec{L})Y_{\ell m} \\
&\quad \frac{1}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}} \int d\Omega Y^*_{\ell' m'} \vec{L} \cdot (\vec{r} \times \vec{L})Y_{\ell m} \\
&\quad \frac{1}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}} \int d\Omega Y^*_{\ell' m'} \vec{r} \cdot (\vec{L} \times \vec{L})Y_{\ell m} = 0.
\end{aligned}$$

because $\vec{L} \times \vec{L} = i\vec{L}$, so $\vec{r} \cdot (\vec{L} \times \vec{L}) = \vec{r} \cdot \vec{L} = 0$.

The justification for claiming $\vec{r}\cdot\vec{E}$ and $\vec{r}\cdot\vec{H}$ satisfy the Helmholtz equation required them to be divergenceless, which $\vec{r}\cdot\vec{E}$ is not in the presence of sources.
The trick is to evaluate $\vec{E}' := \vec{E} + i\vec{J}/\omega\epsilon_0$, so $\vec{r}\cdot\vec{E}'$ and $\vec{r}\cdot\vec{H}$ do satisfy inhomogeneous Helmholtz equations with sources given by ρ and $\vec{J}$, with the latter supplemented by any intrinsic magnetization.
This is somewhat messy, given in section 9.10, but we will not elaborate.

A major use has the sources given by quantum mechanical operators for atomic or nuclear structure, and the vector potential is then a wave function for outgoing photons, giving a decay probabilities rather than radiation power flux. But we will skip this as well, and proceed to discuss scattering of electromagnetic waves.

Scattering of waves

Currents create fields, but fields affect the motion of charges too. Mutual reaction.

Scattering by small scatterers:

Incident wave in direction $\hat{n}_i$: [1]

$$\vec{E}_{\text{inc}} = \vec{\epsilon}_i E_i e^{ik\hat{n}_i\cdot\vec{x}}, \qquad \vec{H}_{\text{inc}} = \hat{n}_i \times \vec{E}_{\text{inc}}/Z_0,$$

with $e^{-i\omega t}$ understood, $k = \omega/c$.

If scatterer is small, its radiation is dominated by dipole terms, electric dipole moment $\vec{p}$ and magnetic dipole moment $\vec{m}$.

Far from scatterer, $r \gg \lambda$,

$$\begin{aligned} \vec{E}_{\text{sc}} &= \frac{k^2 e^{ikr}}{4\pi\epsilon_0 r}\left[(\hat{r}\times\vec{p})\times\hat{r} - \hat{r}\times\vec{m}/c\right]. \\ \vec{H}_{\text{sc}} &= \hat{r}\times\vec{E}_{\text{sc}}/Z_0. \end{aligned}$$

Wave radiates in all directions.

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[1] Notation changes from Jackson: $\vec{\epsilon}_0 \to \vec{\epsilon}_i$, and $_i$ generally for incident wave. His $\vec{n} \to \hat{r}$.

Cross section

Scattering is measured by cross sections.

For classical particle dynamics, $\frac{d\sigma}{d\Omega}$ is the area of the incident beam which gets scattered into the solid angle $d\Omega$.

For wave, $\frac{d\sigma}{d\Omega} = \frac{\text{power scattered into } d\Omega}{\text{incident flux}}$.

Flux $\frac{1}{2}\hat{r}\cdot\left(\vec{E}_{\text{sc}}\times\vec{H}^*_{\text{sc}}\right) = \frac{1}{2Z_0}\hat{r}\cdot\left(\vec{E}_{\text{sc}}\times\left(\hat{r}\times\vec{E}^*_{\text{sc}}\right)\right) = \frac{1}{2Z_0}\vec{E}_{\text{sc}}\cdot\vec{E}^*_{\text{sc}}$, as $\hat{r}\cdot\vec{E}_{\text{sc}} = 0$. But the outgoing wave consists of two polarizations, and we can ask what the cross section is for a given polarization, $\vec{\epsilon}$, for an incident wave with polarization $\vec{\epsilon}_i$. So

$$\frac{d\sigma}{d\Omega}(\hat{r},\vec{\epsilon};\hat{n}_i,\vec{\epsilon}_i) = r^2\frac{\left|\vec{\epsilon}^{\,*}\cdot\vec{E}_{\text{sc}}\right|^2}{\left|\vec{\epsilon}_i^{\,*}\cdot\vec{E}_{\text{inc}}\right|^2},$$

$$\frac{d\sigma}{d\Omega}(\hat{r},\vec{\epsilon};\hat{n}_i,\vec{\epsilon}_i) = \frac{k^4}{(4\pi\epsilon_0 E_i)^2}\left|\vec{\epsilon}^{\,*}\cdot\vec{p} + (\hat{r}\times\vec{\epsilon}^{\,*})\cdot\vec{m}/c\right|^2,$$

where we need to know the $\vec{p}$ and $\vec{m}$ induced by $\vec{E}_{\rm inc}$.

If scatterers are small ($\ll\lambda$), polarization response should be quasi-static, independent of ω, so $\dfrac{d\sigma}{d\Omega}\propto k^4\propto\omega^4$. This is known as Rayleigh's law.

Dielectric Sphere

Last term you found (Jackson 4.56) that a non-magnetic dielectric sphere of radius a has a static induced dipole moment

$$\vec{p} = 4\pi\epsilon_0 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) a^3 \vec{E}_{\text{inc}},$$

and of course $\vec{m} = 0$. So

$$\frac{d\sigma}{d\Omega}(\hat{r}, \vec{\epsilon}; \hat{n}_i, \vec{\epsilon}_i) = k^4 a^6 \left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2 |\vec{\epsilon}^* \cdot \vec{\epsilon}_i|^2 .$$

The scattered wave has the electric field in the plane of the incident polarization and $\hat{r}$; if $\vec{\epsilon}^* \perp \vec{\epsilon}_i$ the amplitude is zero.

If the incident wave is unpolarized, say coming in the z direction, we may take the average over polarization in ϕ, $\vec{\epsilon}_i = (\cos\phi, \sin\phi, 0)$. If we are looking at an angle θ, say with $\hat{r} = (\sin\theta, 0, \cos\theta)$, the two polarization vectors are $\vec{\epsilon}_\| = (\cos\theta, 0, \sin\theta)$ in the scattering plane and $\vec{\epsilon}_\perp = (0, 1, 0)$ perpendicular to it.

Then $\left|\vec{\epsilon}_{\parallel}^{\,*}\cdot\vec{\epsilon}_i\right|^2 = \cos^2\theta\cos^2\phi$ and $|\vec{\epsilon}_{\perp}^{\,*}\cdot\vec{\epsilon}_i|^2 = \sin^2\phi$, with average values (over ϕ of $\frac{1}{2}\cos^2\theta$ and $\frac{1}{2}$ respectively. So

$$\begin{aligned} \frac{d\sigma_{\parallel}}{d\Omega} &= \frac{k^4a^6}{2}\left|\frac{\epsilon_r-1}{\epsilon_r+2}\right|^2\cos^2\theta, \\ \frac{d\sigma_{\perp}}{d\Omega} &= \frac{k^4a^6}{2}\left|\frac{\epsilon_r-1}{\epsilon_r+2}\right|^2. \end{aligned}$$

The *polarization* is defined by the difference over the sum,

$$\Pi(\theta) := \frac{\frac{d\sigma_{\perp}}{d\Omega}-\frac{d\sigma_{\parallel}}{d\Omega}}{\frac{d\sigma_{\perp}}{d\Omega}+\frac{d\sigma_{\parallel}}{d\Omega}} \quad \left(=\frac{\sin^2\theta}{1+\cos^2\theta}\ \text{for dielectric sphere}\right)$$

Unpolarized and Total Cross Sections

If we don't measure the polarization of the scattered light, the unpolarized cross section is the sum of the two,

$$\frac{d\sigma}{d\Omega} = \frac{k^4 a^6}{2} \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2 \left(1 + \cos^2\theta\right)$$

and the total cross section is the integral of this over $d\Omega = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi$,

$$\begin{aligned} \sigma &= \pi k^4 a^6 \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2 \int_{-1}^{1} d(\cos\theta) \left(1 + \cos^2\theta\right) \\ &= \frac{8\pi}{3} k^4 a^6 \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2 . \end{aligned}$$

We will skip subsection C.