Sources of Electromagnetic Fields

Lecture 11 February 28, 2011

We now start to discuss radiation in free space.
We will reorder the material of Chapter 9, bringing sections 6 and 7 up front.
We will also review some of §6.4, on Green functions, which we did (somewhat differently) in the first lecture.

Assume a set of charges with given motion.
What fields do they generate? ($\sim$ wave-guide last time)
We are ignoring back-reaction, the changes in their motion due to the fields. Therefore linearity.
So we can fourier-transform in time.

The Basic Equations

Consider one fourier component, with all fields having a time dependence $e^{-i\omega t}$.
They are generated by the fourier components

$$\begin{aligned} \rho(\vec{x}, t) &= \rho(\vec{x}) e^{-i\omega t}, \\ \vec{J}(\vec{x}, t) &= \vec{J}(\vec{x}) e^{-i\omega t}. \end{aligned}$$

The fields $\vec{E}(\vec{x}, t)$ and $\vec{H}(\vec{x}, t)$ generated by these charges and currents may be described by the electrostatic potential $\Phi(\vec{x}, t)$ and the vector potential $\vec{A}(\vec{x}, t)$.
But as discussed in §6.2, Φ and $\vec{A}$ have a gauge-invariance. To determine them, need Maxwell's equations *and* a gauge condition.

Choose *Lorenz Gauge*: $\vec{\nabla} \cdot \vec{A} + \dfrac{1}{c^2} \dfrac{\partial \Phi}{\partial t} = 0$. Then the fields $\vec{A}$ and Φ are determined.

Then from §6.2,

$$\begin{aligned} \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} &= -\rho/\epsilon_0, \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{J}. \end{aligned}$$

For our fourier mode, $\partial/\partial t \to -i\omega$, so each component of $\vec{A}$ and Φ satisfy

$$\left(\nabla^2 + k^2\right)\Psi(\vec{x}) = -s(\vec{x}), \tag{1}$$

with $k = \omega/c$, and with the prescribed source $s(\vec{x})$.

Solution by Green function, from §6.4. Let us review this.

Solution by Green function

$\left(\nabla^2 + k^2\right)\Psi(\vec{x}) = -s(\vec{x})$ is (inhomogeneously) linear in Ψ, so solution is sum of solutions of the homogeneous Helmholtz equation and specific solutions for "each piece" of the source. The equation is an *elliptic partial differential equation*, having a unique solution once boundary conditions are specified.

Think of the source as a sum of pieces at each point,

$$s(\vec{x}) = \int d\vec{x}\,' s(\vec{x}\,')\delta^3(\vec{x}\,' - \vec{x}),$$

and solve for a delta function source with the Green function

$$\left(\nabla_x^2 + k^2\right) G(\vec{x}, \vec{x}\,') = -\delta^3(\vec{x} - \vec{x}\,'). \qquad (2)$$

Then the solution for Ψ is

$$\Psi(\vec{x}) = \int d^3\vec{x}\,'\, s(\vec{x}\,')G(\vec{x}, \vec{x}\,').$$

Finding $G(\vec{x}, \vec{x}')$

A point charge at origin: $V(\vec{x}) = \dfrac{q}{4\pi\epsilon_0 |x|}$

$\vec{E} = -\vec{\nabla} V = q\dfrac{\vec{x}}{4\pi\epsilon_0 |x|^3}$, $\vec{\nabla}\cdot\vec{E} = -\nabla^2 V = q\delta^3(\vec{x})$.

So we see $\phi(\vec{x}) = \dfrac{1}{|x|}$ has $\vec{\nabla}\phi = -\dfrac{\vec{x}}{|x|^3}$, $\nabla^2\phi = -4\pi\delta^3(\vec{x})$.

On the other hand, $W := e^{\pm ik|x|}$ satisfies $\vec{\nabla} W = \pm ik\dfrac{\vec{x}}{|x|}W$

and

$$\begin{aligned}
\nabla^2 W &= \left[\pm ik\left(\frac{\vec{\nabla}\cdot\vec{x}}{|x|} - \vec{x}\cdot\vec{\nabla}\frac{1}{|x|}\right)W - k^2\frac{\vec{x}^{\,2}}{|x|^2}\right]W \\
&= \left[\pm ik\left(\frac{3}{|x|} - \frac{\vec{x}^{\,2}}{|x|^3}\right) - k^2\right]W = \pm\frac{2ik}{|x|}W - k^2 W.
\end{aligned}$$

Thus

$$\begin{aligned}
\left(\nabla_x^2 + k^2\right) W\phi &= \left(\nabla_x^2 W\right)\phi + 2(\vec{\nabla} W)\cdot(\vec{\nabla}\phi) + W\left(\nabla_x^2\phi\right) \\
&\quad + k^2 W\phi
\end{aligned}$$

$$
\begin{aligned}
\left(\nabla_x^2 + k^2\right) W\phi &= \left(\nabla_x^2 W\right)\phi + 2(\vec{\nabla}W)\cdot(\vec{\nabla}\phi) + W\left(\nabla_x^2\phi\right) \\
&\qquad + k^2 W\phi \\
&= \pm\frac{2ik}{|x|}W\phi \mp 2ik\frac{\vec{x}}{|x|}W\cdot\frac{\vec{x}}{|x|^3} - 4\pi W\delta^3(\vec{x}) \\
&= -4\pi\delta^3(\vec{x})
\end{aligned}
$$

as $W(\vec{x})\delta^3(\vec{x}) = W(\vec{0})\delta^3(\vec{x}) = \delta^3(\vec{x})$.

So for $\vec{x}\,' = 0$, $G(\vec{x},\vec{0}) = \dfrac{W\phi}{4\pi} = \dfrac{e^{\pm ik|x|}}{4\pi|x|}$.
But the operator $\left(\nabla_x^2 + k^2\right)$ is translation-invariant, so we may translate:

$$
G(\vec{x},\vec{x}\,') = \frac{e^{\pm ik|\vec{x}-\vec{x}\,'|}}{4\pi|\vec{x}-\vec{x}\,'|}. \qquad (3)
$$

We've ignored boundary conditions. Want outgoing waves only. Choose upper sign.

In Spherical Coordinates

We are often interested in sources confined to a compact area and how it radiates out to large distances. Spherical coordinates are most suitable.

In spherical coordinates $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}L^2$ and $\delta^3(\vec{x} - \vec{x}\,') = \frac{1}{r^2 \sin\theta}\delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi')$ due to the metric factors h_i, From the completeness relation (J3.56) for the spherical harmonics, the angular part of the delta function can be written as a sum, and we have

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}L^2 + k^2\right) G(\vec{x}, \vec{x}\,')$$

$$= -\frac{\delta(r-r')}{r^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi),$$

Writing $G(\vec{x}, \vec{x}\,') = \sum_{\ell m} R_{\ell m}(r, \vec{x}\,') Y_{\ell m}(\theta, \phi)$, we have

$$\sum_{\ell m} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} + k^2 \right) R_{\ell m}(r, \vec{x}\,') Y_{\ell m}(\theta, \phi)$$

$$= -\frac{1}{r^2}\delta(r-r') \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi),$$

so we see that $R(r, \vec{x}\,') = \sum_\ell g_\ell(r, r') Y^*_{\ell m}(\theta', \phi')$ where

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} + k^2 \right) g_\ell(r, r') = -\frac{1}{r^2}\delta(r - r').$$

For $r \neq r'$ this is just the spherical Bessel equation, so the solutions are combinations of $j_\ell(kr)$ and $n_\ell(kr)$, or better of $j_\ell(kr)$ and $h^{(1)}_\ell := j_\ell(kr) + i n_\ell(kr) \underset{r \gg \ell}{\longrightarrow} (-i)^{\ell+1} e^{ikr}/kr$.
For $r < r'$ we need the solution to be regular at $r = 0$, so there are no n or h contributions, only j_ℓ,

$$g_\ell(r, r') = a_\ell(r') j_\ell(kr) \qquad \text{for } r < r',$$

while for $r > r'$ we want only outgoing waves, with e^{+ikr}, so the solution is pure $h_\ell^{(1)}$ with no $h_\ell^{(2)}$ (or j_ℓ)

$$g_\ell(r, r') = b_\ell(r') h_\ell^{(1)}(kr) \qquad \text{for } r > r'.$$

But from (3) we see that the Green's function is symmetric under $\vec{x} \leftrightarrow \vec{x}\,'$,

$$G(\vec{x}, \vec{x}\,') = \frac{e^{\pm ik|\vec{x}-\vec{x}\,'|}}{4\pi|\vec{x}-\vec{x}\,'|} \quad (3)$$

so $a_\ell(r) = a_\ell h_\ell^{(1)}(kr)$ and $b_\ell(r) = a_\ell j_\ell(kr)$, and we may write more generally

$$g_\ell(r, r') = a_\ell j_\ell(kr_<) h_\ell^{(1)}(kr_>),$$

where $r_<$ is the smaller of r and r' and $r_>$ is the greater.

The delta function source for the spherical Bessel equation on $g_\ell(r,r')$ means the first derivative is discontinuous,

$$g'_\ell(r{=}r'{+}\epsilon) - g'_\ell(r{=}r'{-}\epsilon) = -\frac{1}{r'^{\,2}}$$
$$= a_\ell k j_\ell(kr) h'^{(1)}_\ell(kr) - a_\ell k h^{(1)}_\ell(kr) j'_\ell(kr).$$

This is ka_ℓ times the Wronskian of $h^{(1)}_\ell$ and j_ℓ, which should be $-r'^{\,-2}$. This agrees with the general statement that the Wronskian satisfies $dW/dr = -P(r)W$, where $P(r)$ is the coefficient of the first order term, here $2/r$. Thus we can determine a_ℓ at any point r', and as Jackson 3.89-90 tells us for small x,

$$\begin{aligned} j_\ell(x) &= \sqrt{\frac{\pi}{2x}}\,J_{\ell+1/2}(x) \to \frac{\sqrt{\pi}}{2\Gamma(\ell+3/2)}\left(\frac{x}{2}\right)^\ell, \\ n_\ell(x) &= \sqrt{\frac{\pi}{2x}}\,N_{\ell+1/2}(x) \to -\Gamma(\ell+1/2)\left(\frac{2}{x}\right)^{\ell+1}\frac{1}{2\sqrt{\pi}}, \\ h^{(1)}_\ell &= j_\ell + i n_\ell \to i n_\ell, \end{aligned}$$

so $j_\ell(r)h'^{(1)}_\ell(kr) - h^{(1)}_\ell(r)j'_\ell(kr) \to i/(kr)^2$, and $a_\ell = ik$.

Green function in spherical coordinates

So all together

$$\begin{aligned} G(\vec{x} - \vec{x}\,') &= \frac{e^{ik|\vec{x}-\vec{x}\,'|}}{4\pi|\vec{x} - \vec{x}\,'|} \\ &= ik \sum_{\ell m} j_\ell(kr_<) h_\ell^{(1)}(kr_>) Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi). \end{aligned}$$

We are now ready to examine the solutions to the Helmholtz equation,

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \vec{J}(\vec{x}\,') \frac{e^{ik|\vec{x}-\vec{x}\,'|}}{|\vec{x} - \vec{x}\,'|}$$

$$= i\mu_0 k \sum_{\ell m} \int d^3x' j_\ell(kr_<) h_\ell^{(1)}(kr_>) Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi) \vec{J}(\vec{x}\,').$$

If the sources are restricted to some region $|\vec{x}\,'| < d$, and we are asking about positions further from the origin, $r > d$, then $r_< = r'$ and $r_> = r$, and

$$\vec{A}(\vec{x}) = i\mu_0 k \sum_{\ell m} h_\ell^{(1)}(kr) Y_{\ell m}(\theta, \phi) \int d^3x' j_\ell(kr') Y^*_{\ell m}(\theta', \phi') \vec{J}(\vec{x}\,').$$

We see that $\vec{A}$ has an expansion in specified modes (ℓ, m) with the sources only determining the coefficients of these modes. If the source region is small compared to the wavelength, $d \ll \lambda = 2\pi/k = 2\pi c/\omega$, we have $kr' \ll 1$ whereever $\vec{J}(\vec{x}\,') \neq 0$, so we may use the expansion $j_\ell(x) \approx x^\ell/(2\ell+1)!!$, appropriate for $x \ll 1$. We see that the lowest ℓ value which contributes will dominate.

Zones (Jackson §9.1)

We have derived a quite general expression for the fields, but we can almost always find ourselves in a zone for which things simplify, depending on the relative sizes of d, λ, and r.

If d and r are both much smaller than λ, we are in the *near* zone, we may set $k = 0$ while setting $kj_\ell(kr_<)h_\ell^{(1)}(kr >)$ to $\frac{-i}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}$. The fields are essentially instantaneously generated by the currents and charges. If, in addition we assume $d \ll r$, the lowest ℓ value will dominate.

If $r \gg \lambda$ and $r > d$, the fields oscillate rapidly with $h_\ell^{(1)}(kr) \to (-i)^{\ell+1}e^{ikr}/r$, falling off only as $1/r$, typical of radiation fields, and this is called the *far* or *radiation* zone. If we also have $d \ll \lambda$, $kr_<$ is small whereever $\vec{J}$ doesn't vanish, and the lowest ℓ mode will dominate.

We have not bothered to find $\Phi(\vec{x})$ because the Lorenz gauge $-i\omega\Phi/c^2 = -\vec{\nabla}\cdot\vec{A}$ gives it in terms of $\vec{A}$, except for $\omega = 0$, for which $\Phi(\vec{x})$ is given by the static Coulomb expression integrated over all the charges, the electric field is given by Coulombs law, and there is no magnetic field arising from Φ.

Electric Dipole

Now let us consider the zone $d \ll \lambda \ll r$, which should be dominated by the lowest ℓ. If the $\ell = 0$ term does not vanish, we may write

$$\begin{aligned}\vec{A}(\vec{x}) &\approx i\mu_0 k h_0^{(1)}(kr) Y_{00} \int d^3x' Y_{00}^* \vec{J}(\vec{x}') \\ &= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \vec{J}(\vec{x}'),\end{aligned}$$

because $h_0^{(1)}(x) = -ie^{ix}/x$. We are assuming all sources have an $e^{-i\omega t}$ time dependence, so the continuity equation tells us $\vec{\nabla} \cdot \vec{J} = -\partial\rho/\partial t = i\omega\rho$, we may write[1] $\int d^3x' \vec{J}(\vec{x}') = -i\omega \int d^3x' \, \vec{x}' \rho(\vec{x}')$. The integral is just the electric dipole moment, so $\vec{A}(\vec{x}) \approx -\dfrac{i\mu_0\omega}{4\pi} \vec{p} \dfrac{e^{ikr}}{r}$.

[1]See the lecture notes for some algebra justifying this and the messy expression for $\vec{E}$ below.

That expression, $\vec{A}(\vec{x}) \approx -i\mu_0\omega\,\vec{p}\,e^{ikr}/4\pi r$, is accurate for all $r > d$ to lowest order in d/λ, provided the dipole moment isn't zero.
Quite generally, $\vec{H} = \dfrac{1}{\mu_0}\vec{\nabla}\times\vec{A}$, while *outside the region with sources*,

$$\frac{\partial\vec{D}}{\partial t} = -i\omega\epsilon_0\vec{E} = \vec{\nabla}\times\vec{H} \Longrightarrow \vec{E} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H},$$

with $Z_0 = \sqrt{\mu_0/\epsilon_0}$. The curl of $\vec{p}f(r)$ is $\hat{r}\times\vec{p}\,\partial f/\partial r$, so for our electric dipole source, we have[2]

$$\vec{H} = \frac{ck^2}{4\pi}\,\hat{r}\times\vec{p}\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r},$$

and

$$\vec{E}(\vec{x}) = \frac{e^{ikr}}{4\pi\epsilon_0 r}\left\{-k^2\hat{r}\times(\hat{r}\times\vec{p}) + [3\hat{r}(\hat{r}\cdot\vec{p}) - \vec{p}]\left(\frac{1}{r^2} - \frac{ik}{r}\right)\right\}.$$

[2] See lecture notes for algebra details.

Note the first term in $\vec{E}$ is perpendicular to $\vec{x}$, but the second is not. However this longitudinal term falls off as r^{-2}, so may be neglected in the radiation zone $r \gg \lambda$, where we can write

$$\left.\begin{aligned} \vec{H} &= \frac{ck^2}{4\pi}\,\hat{r}\times\vec{p}\,\frac{e^{ikr}}{r} \\ \vec{E} &= \frac{-k^2e^{ikr}}{4\pi\epsilon_0 r}\hat{r}\times(\hat{r}\times\vec{p}) = -Z_0\hat{r}\times\vec{H} \end{aligned}\right\} \quad \text{in the } \textbf{radiation zone}$$

In the *near zone*, that is when $d < r \ll \lambda$, we have

$$\left.\begin{aligned} \vec{H} &= \frac{i\omega}{4\pi r^2}\hat{r}\times\vec{p} \\ \vec{E} &= \frac{1}{4\pi\epsilon_0 r^3}\left(3\hat{r}(\hat{r}\cdot\vec{p}) - \vec{p}\right) \end{aligned}\right\} \quad \text{in the } \textbf{near zone}$$

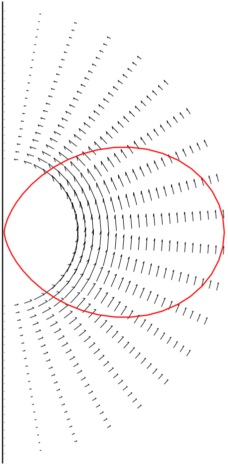
The electric field in the near zone is just what we would have from a static dipole of the present value at each moment, and the E field dominates the H field in this zone.

Power radiated at large distances:
average power per unit solid angle is

$$\begin{aligned}\frac{\langle P\rangle}{d\Omega} &= \frac{r^2}{2}\mathrm{Re}\,\hat r\cdot(\vec E\times\vec H^*) \approx \frac{Z_0c^2k^4}{2(4\pi)^2}\left|\hat r\times(\hat r\times\vec p)\right|^2\\ &= \frac{Z_0c^2k^4}{32\pi^2}\,p^2\,(1-\cos^2\theta) = \frac{Z_0c^2k^4}{32\pi^2}\,p^2\,\sin^2\theta,\end{aligned}$$

where θ is the angle between $\vec p$ and $\vec x$. The total power radiated is

$$\begin{aligned}\langle P\rangle &= 2\pi\int_0^\pi d\theta\,\sin\theta\,\frac{\langle P\rangle}{d\Omega}\\ &= \frac{Z_0c^2k^4}{16\pi}\,p^2\int_0^\pi d\theta\,\sin^3\theta\\ &= \frac{Z_0c^2k^4}{12\pi}\,p^2.\end{aligned}$$



Next order, $\ell = 1$

If $\ell = 0$ vanishes, need $\ell = 1$ term in the expansion,

$$\vec{A}^{(1)} = i\mu_0 k h_1^{(1)}(kr) \sum_{m=-1}^{m=1} Y_{1m}(\theta,\phi) \int d^3x' j_1(kr') Y_{1m}^*(\theta',\phi') \vec{J}(\vec{x}\,').$$

With

$$h_1^{(1)}(x) = -\frac{e^{ix}}{x}\left(1 + \frac{i}{x}\right), \quad j_1(x) = \frac{x}{3}\left(1 + \mathcal{O}x^2\right),$$

and

$$\sum_{m=-1}^{m=1} Y_{1m}(\theta,\phi) Y_{1m}^*(\theta',\phi') = \frac{3}{4\pi}\hat{r}\cdot\hat{r'},$$

we see that [3]

$$\vec{A}^{(1)} = i\mu_0 k \frac{3}{4\pi}\frac{1}{3}\frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)\int d^3x' \hat{r}\cdot\vec{x}\,'\vec{J}(\vec{x}\,').$$

[3] I disagree with Jackson 9.30 by an overall sign

The multipole moments involved here are tensors $\sim \vec{x}\,'\vec{J}(\vec{x}\,')$. The antisymmetric part is the integral of the magnetization

$$\mathcal{M}(\vec{x}\,') = \frac{1}{2}\vec{x}\,' \times \vec{J}(\vec{x}\,'), \quad \text{with } \vec{m} = \int d^3x' \mathcal{M}(\vec{x}\,')$$

the magnetic dipole moment.

$$\hat{r} \times \vec{m} = \frac{1}{2}\int d^3x' \left[(\hat{r}\cdot\vec{J}(\vec{x}\,')\vec{x}\,' - (\hat{r}\cdot\vec{x}\,')\vec{J}(\vec{x}\,')\right].$$

The symmetric piece is related to the electric quadripole moment

$$\begin{aligned}
Q_{ij} &:= \int d^3x' \left(3x'_i x'_j - x'^{\,2}\delta_{ij}\right)\rho(\vec{x}\,') \\
&= \int d^3x' \left(3x'_i x'_j - x'^{\,2}\delta_{ij}\right)\frac{-i}{\omega}\vec{\nabla}\cdot\vec{J} \\
&= \frac{i}{\omega}\int d^3x' J_k(\vec{x}\,')\partial'_k \left(3x'_i x'_j - x'^{\,2}\delta_{ij}\right) \\
&= \frac{i}{\omega}\int d^3x' J_k(\vec{x}\,') \left(3\delta_{ik}x'_j + 3\delta_{jk}x'_i - 2x'_k\delta_{ij}\right)
\end{aligned}$$

Again, $Q_{ij} = \frac{i}{\omega} \int d^3x' \left(3x'_j J_i + 3x'_i J_j - 2(\vec{x}\,' \cdot J)\delta_{ij}\right)$.
So $\hat{r} \cdot Q = \sum \hat{r}_i Q_{ij} \hat{e}_j =$
$\frac{i}{\omega} \int d^3x' \left(3\hat{r} \cdot \vec{J}(\vec{x}\,')\,\vec{x}\,' + 3\hat{r} \cdot \vec{x}\,'\,\vec{J}(\vec{x}\,') - 2\vec{x}\,' \cdot \vec{J}(\vec{x}\,')\,\hat{r}\right)$.

For completeness we need to consider a electric monopole term

$$M_E = \int d^3x' x'^2 \rho(\vec{x}\,') = \frac{2i}{\omega} \int d^3x' \vec{x}\,' \cdot \vec{J}.$$

So our complete $\ell = 1$ vector potential is

$$\vec{A}^{(1)} = -i \frac{\mu_0 k}{24\pi} \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr}\right) (6\hat{r} \times \vec{m} + i\omega\hat{r} \cdot \mathbf{Q} + i\omega M_E \hat{r}).$$

Let us evaluate $\vec{H}$ and $\vec{E}$ only to leading order in $1/r$, so we need only consider the derivative acting on e^{ikr}, and needn't worry about $\vec{\nabla} \times \hat{r}$. We can also drop the i/kr term.

Magnetic Dipole

Then $\vec{H}^{(1)} = \frac{1}{\mu_0}\vec{\nabla} \times \vec{A}^{(1)}$

$$= \frac{k^2}{24\pi}\frac{e^{ikr}}{r}\hat{r} \times (6\hat{r} \times \vec{m} + i\omega\hat{r} \cdot \mathbf{Q} + i\omega M_E \hat{r}).$$

Notice that the electric monopole vanishes due to $\hat{r} \times \hat{r} = 0$. The magnetic dipole contributes

$$\begin{aligned}
\vec{H}^{\mathrm{MD}} &= \frac{k^2}{4\pi}\frac{e^{ikr}}{r}\hat{r} \times (\hat{r} \times \vec{m}), \\
\vec{E}^{\mathrm{MD}} &= \frac{iZ_0}{k}\vec{\nabla} \times \vec{H}^{\mathrm{MD}} = -\frac{k^2 Z_0}{4\pi}\frac{e^{ikr}}{r}\hat{r} \times (\hat{r} \times (\hat{r} \times \vec{m})) \\
&= \frac{k^2 Z_0}{4\pi}\frac{e^{ikr}}{r}\,\hat{r} \times \vec{m}.
\end{aligned}$$

These are of the same form as for the electric dipole, but with $\vec{E}$ and $\vec{H}$ interchanged. The radiation pattern is the same, but the polarization has $\vec{E} \perp \vec{m}$ here, while $\vec{E}$ lies in the plane including $\hat{r}$ and $\vec{p}$ for the electric dipole.

Electric Quadripole

One might be tempted to think the electric quadripole also vanishes, as it involves $\hat{r} \times (\hat{r} \cdot \mathbf{Q})$, and $\mathbf{Q}$ is symmetric. But that is incorrect: in $\hat{r} \times (\hat{r} \cdot \mathbf{Q}) = \sum \epsilon_{ijk} \hat{r}_i Q_{j\ell} \hat{r}_\ell \hat{e}_k$, the summands are symmetric under $i \leftrightarrow \ell$ and antisymmetric under $i \leftrightarrow j$, but that does not make things vanish. Jackson defines the vector $\vec{Q}(\vec{n}) := \sum Q_{ij} n_j \hat{e}_i$, and then we have $\hat{r} \times \vec{Q}(\hat{r})$. Then again keeping only $1/r$ terms,

$$\begin{aligned}
\vec{H}^{\mathrm{EQ}} &= \frac{ick^3}{24\pi} \frac{e^{ikr}}{r} \hat{r} \times \vec{Q}(\hat{r}) \\
\vec{E}^{\mathrm{EQ}} &= \frac{-iZ_0 ck^3}{24\pi} \frac{e^{ikr}}{r} \hat{r} \times \left(\hat{r} \times \vec{Q}(\hat{r}) \right).
\end{aligned}$$

Power radiated

Probably the most interesting thing one might ask is how much power is radiated, and in which directions, as we did for the electric dipole.

For an electric quadripole, $|\mathbf{Q}|$ is a symmetric real traceless tensor, so we could rotate the coordinate system so that it will be diagonal. If we take an axially symmetric case, with $Q_{zz} = -2Q_{xx} > 0$. The average power per unit solid angle is

$$\begin{aligned} \frac{\langle P \rangle}{d\Omega} &= \frac{r^2}{2} \mathrm{Re}\, \hat{r} \cdot (\vec{E} \times \vec{H}^*) \\ &\propto \left| \hat{r} \times (\hat{r} \times \mathbf{Q}(\hat{r})) \right|^2 \end{aligned}$$

as shown.