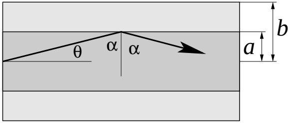
Fiber Optics

Waves can be guided not only by conductors, but by dielectrics. Fiber optics cable of silica has $n(r)$ varying with radius.

Simplest: core radius a with $n = n_1$, surrounded (radius b) with $n = n_0 < n_1$. Total internal reflection if



$$\alpha > \alpha_c = \sin^{-1}(n_0/n_1)$$

Equivalently $\theta < \theta_{\max} = \cos^{-1}(n_0/n_1)$. This is geometrical optics. Needs $\lambda \ll a$.

Two kinds of fibers:

- multimode, $a \gg \lambda$, treat with geometrical optics. Typically $a \approx 25\,\mu\text{m}$, $b \approx 75\,\mu\text{m}$, $\lambda \sim 0.85\,\mu\text{m}$.
- single mode, $a \sim \lambda$, treat as wave guide. Typically $a \approx 2\,\mu\text{m}$.

Multimode fibers

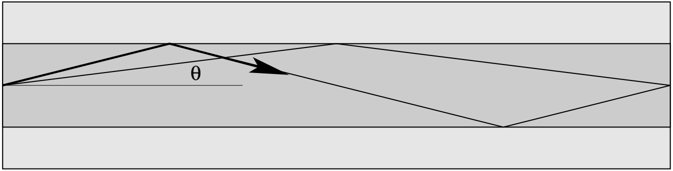
Define $\Delta = \dfrac{n_1^2 - n_0^2}{2n_1^2} \approx 1 - \dfrac{n_0}{n_1}$, typically about 0.01.
$\cos\theta_{\max} \approx 1 - \frac{1}{2}\theta_{\max}^2 = 1 - \Delta$, so $\theta_{\max} \approx \sqrt{2\Delta}$.
How many modes can propagate?
Uncertainty principle: only one mode can fit per unit "volume" in phase space, $N = \displaystyle\int \left(\frac{dpdq}{2\pi\hbar}\right)^D$ for each mode in D dimensions. Here $D = 2$, the cross section has coordinate integral $\int d^2q = \pi a^2$. As
$|\vec{k}_\perp| \le k_z \tan\theta_{\max} = k_z\sqrt{2\Delta}$,
$\int d^2p = \hbar^2 \int d^2k = 2\pi\hbar^2 k_z^2\Delta$. There are two polarizations, so

$$N = 2\frac{1}{(2\pi)^2}\left(\pi a^2\right)\left(2\pi k_z^2\Delta\right) = \frac{1}{2}V^2,$$

where$V := ka\sqrt{2\Delta}$ is called the *fiber parameter*.

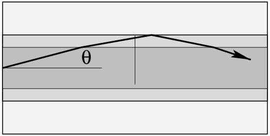
Problem with simple fiber

At each angle $\theta < \theta_{\text{max}}$, light travels indefinitely down the fiber. But to go a large distance L down the fiber,



it travels a different distance $L \sec\theta$, so light from different θ's arrive with different phases, and interfere!

Fix: make several transitions to lower n. In fact, for homework (Jackson 8.14) you will find a "perfect" fix, using n varying continuously with radius.



$n(x)$ varying with radius x

Consider dielectric with $\epsilon(\vec{x})$ varying smoothly, $\mu = \mu_0$ as silica is not magnetic. Assume single frequency ω.
Maxwell gives

$$\begin{aligned}
\vec{\nabla} \cdot \epsilon\vec{E} &= 0 = \left(\vec{\nabla}\epsilon\right) \cdot \vec{E} + \epsilon\vec{\nabla} \cdot \vec{E} \\
\vec{\nabla} \times \vec{E} &= -\mu_0 \frac{\partial \vec{H}}{\partial t} = i\mu_0\omega\vec{H} \\
\vec{\nabla} \times \vec{H} &= \frac{\partial \epsilon\vec{E}}{\partial t} = -i\omega\epsilon\vec{E} \\
\vec{\nabla} \cdot \vec{H} &= 0.
\end{aligned}$$

So

$$\begin{aligned}
\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) &= -\nabla^2\vec{E} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) = i\mu_0\omega\vec{\nabla} \times \vec{H} \\
&= \mu_0\omega^2\epsilon\vec{E} \\
&= -\nabla^2\vec{E} - \vec{\nabla}\left(\frac{1}{\epsilon}\left(\vec{\nabla}\epsilon\right) \cdot \vec{E}\right)
\end{aligned}$$

The same for H:

$$\begin{aligned}
\vec{\nabla} \times \left(\vec{\nabla} \times \vec{H}\right) &= -\nabla^2 \vec{H} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{H}\right) = -i\omega \vec{\nabla} \times \left(\epsilon \vec{E}\right) \\
-\nabla^2 \vec{H} &= -i\omega \left(\vec{\nabla}\epsilon\right) \times \vec{E} - i\omega\epsilon \vec{\nabla} \times \vec{E} \\
&= -i\omega \left(\vec{\nabla}\epsilon\right) \times \vec{E} + \mu_0 \omega^2 \epsilon \vec{H}.
\end{aligned}$$

Thus

$$\begin{aligned}
\nabla^2 \vec{E} + \mu_0 \omega^2 \epsilon \vec{E} + \vec{\nabla}\left(\frac{1}{\epsilon}\left(\vec{\nabla}\epsilon\right) \cdot \vec{E}\right) &= 0 \\
\nabla^2 \vec{H} + \mu_0 \omega^2 \epsilon \vec{H} - i\omega \left(\vec{\nabla}\epsilon\right) \times \vec{E} &= 0
\end{aligned}$$

Assume ϵ varies slowly compared to λ,

$$\nabla\epsilon \ll \frac{\epsilon}{\lambda} = \frac{\epsilon\omega}{c}.$$

Other terms are ω^2/c^2 times E or H, but $\nabla\epsilon$ terms are $\nabla\epsilon/\epsilon\lambda$ times E, $\ll \lambda^2 = \omega^2/c^2$, so they can be ignored. Both $\vec{E}$ and $\vec{H}$ satisfy

$$\left(\nabla^2 + \frac{\omega^2}{c^2} n^2(\vec{r})\right) \psi(\vec{r}) = 0.$$

Eikonal

ψ oscillates rapidly (on scale $\sim \lambda$). Take this away by defining the eikonal $S(\vec{r})$, with

$$\psi(\vec{r}) = e^{i\omega S(\vec{r})/c}$$

$$\begin{aligned} \text{so} \quad \nabla^2\psi &= \vec{\nabla} \cdot \left(\frac{i\omega}{c} \vec{\nabla} S e^{i\omega S(\vec{r})/c} \right) \\ &= \left[\frac{i\omega}{c} \nabla^2 S - i \left(\frac{\omega}{c} \right)^2 \left(\vec{\nabla} S \right)^2 \right] e^{i\omega S(\vec{r})/c} \\ &= -(\omega^2 n^2 / c^2) e^{i\omega S/c} \end{aligned}$$

and $n^2(\vec{r}) - \vec{\nabla} S \cdot \vec{\nabla} S = -i \dfrac{c}{\omega} \nabla^2 S$. Now $c/\omega \sim \lambda$ while ∇S varies with $n(\vec{r})$, much more slowly, so we can set the r.h.s to zero, for the

Eikonal approximation: $\quad \vec{\nabla} S \cdot \vec{\nabla} S = n^2(\vec{r})$.

The *Eikonal approximation*: $\vec{\nabla}S \cdot \vec{\nabla}S = n^2(\vec{r})$ doesn't give the direction S changes, but does give the rate.

Define $\hat{k}(\vec{r})$ so $\vec{\nabla}S = n(\vec{r})\hat{k}(\vec{r})$. Near a point r_0,

$$\begin{aligned}\psi(\vec{r}) &\approx e^{i\omega\left(S(\vec{r}_0) + (\vec{r} - \vec{r}_0)\cdot\vec{\nabla}S\right)/c} \\ &= e^{i\omega S(\vec{r}_0)/c}\, e^{i\omega\hat{k}\cdot(\vec{r}-\vec{r}_0)n(\vec{r})/c},\end{aligned}$$

so it is locally a plane wave with $|\vec{k}| = \omega n(\vec{r})/c$.
Consider an integral curve, that is, a ray following $\vec{\nabla}S$, and let s be the distance along that curve. Then $d\vec{r}/ds = \hat{k}$, $n(\vec{r})d\vec{r}/ds = \vec{\nabla}S$, so

$$\frac{d}{ds}\left(n(\vec{r})\frac{d\vec{r}}{ds}\right) = \frac{d}{ds}\vec{\nabla}S = \vec{\nabla}\left.\frac{dS}{ds}\right|_{\Gamma} = \vec{\nabla}n(\vec{r}). \qquad (1)$$

Meridional rays pass through axis ($m = 0$ as waves)
skew rays do not, travel helically ($m \neq 0$ modes).

We will treat only meridional, effectively in xz plane, with x a radial direction and z along the fiber. We assume $n(\vec{r}) = n(x)$ independent of z.
Take x and z components of

$$(1): \quad \frac{d}{ds}\left(n(\vec{r})\frac{d\vec{r}}{ds}\right) = \vec{\nabla} n(\vec{r})$$

$$\frac{d}{ds}\left(n(x)\sin\theta\right) = \frac{dn(x)}{dx}, \qquad \frac{d}{ds}\left(n(x)\cos\theta\right) = \frac{dn(\vec{r})}{dz} = 0.$$

So $n(x)\cos\theta =$ constant. With $\theta(0) < \theta_{\text{max}}$ ray reaches a maximum radius x_{max} with $\bar{n} := n(0)\cos\theta(0) = n(x_{\text{max}})$.

$$\frac{dz}{ds} = \cos\theta = \frac{\bar{n}}{n(x)}, \qquad \frac{d}{ds} = \frac{\bar{n}}{n(x)}\frac{d}{dz},$$

so the x component of (1) gives

$$\frac{dn}{dx} = \frac{\bar{n}}{n(x)}\frac{d}{dz}\left(n(x)\frac{\bar{n}}{n(x)}\frac{dx}{dz}\right) = \frac{\bar{n}}{n(x)}\frac{d}{dz}\left(\bar{n}\frac{dx}{dz}\right),$$

$$\text{so} \qquad \bar{n}^2\frac{d^2x}{dz^2} = n(x)\frac{dn(x)}{dx} = \frac{1}{2}\frac{d}{dx}n^2(x).$$

$$\bar{n}^2 \frac{d^2x}{dz^2} = \frac{1}{2}\frac{d}{dx} n^2(x)$$

Looks like $ma = -dV/dx$ with potential $-\frac{1}{2}n^2(x)$ and time z, so as for Newton, multiply by "velocity" dx/dz, to get

$$\frac{1}{2}\bar{n}^2 \frac{d}{dz}\left(\frac{dx}{dz}\right)^2 = \frac{1}{2}\frac{d}{dz} n^2(x) \Longrightarrow \bar{n}^2 \underbrace{\left(\frac{dx}{dz}\right)^2}_{=0 \text{ at } x_{\max}} = n^2(x) - \bar{n}^2.$$

The distance travelled along z in getting from the axis to x is

$$z(x) = \int_0^x \frac{dz}{dx} dx = \bar{n}\int_0^x \frac{dx}{\sqrt{n^2(x) - \bar{n}^2}},$$

and the distance from one axis crossing to the next is

$$Z = 2\bar{n}\int_0^{x\max} \frac{dx}{\sqrt{n^2(x) - \bar{n}^2}}.$$

The optical distance $\int n(x)ds$ between axis crossings is

$$
\begin{aligned}
L_{\text{opt}} &= 2\int_0^{x\text{max}} n(x)\frac{ds}{dz}\frac{dz}{dx}\,dx \\
&= 2\int_0^{x\text{max}} n(x)\frac{n(x)}{\bar{n}}\frac{\bar{n}}{\sqrt{n^2(x)-\bar{n}^2}}\,dx \\
&= 2\int_0^{x\text{max}} \frac{n^2(x)}{\sqrt{n^2(x)-\bar{n}^2}}\,dx.
\end{aligned}
$$

Over a long distance L, many axis crossings (L/Z), total phase change is proportional to $L\frac{L_{\text{opt}}}{Z}$. It is ideal if $\frac{L_{\text{opt}}}{Z}$ is independent of $\bar{n}$, for otherwise different rays will destructively interfere.
You will find the ideal in problem 8.14.
Signals will also degrade with distance if there is dispersion over the bandwidth of the signal. There is also some absorption in real dielectrics. These two issues for silica favor using $\lambda \sim 1.4\mu$m.

We will not cover single-mode fibers, or normal modes in fibers. So we are skipping section 8.11.

Now we will discuss sources of electromagnetic fields in conducting waveguides.

Next time, we will begin discussing sources more generally, We will first cover spherical waves of Jackson §9.6, and then come back to the beginning of chapter 9.

Sources of Waves in Wave Guides

We discussed waves propagation without sources in waveguides. Now consider a given distribution of charges and currents in a localized region of the waveguide. We assume the charge motion is otherwise determined, ignoring back reaction of the fields on the charges.

Then Maxwell's equations are still linear (inhomogeneous) in the fields, with boundary conditions still time-independent, so fourier transform in time will give frequency components independently in terms of frequency components of the source distribution.

Away from the sources, waves as before, with the general fields superpositions of normal modes with z dependence given by

$$k = \pm\sqrt{\omega^2/c^2 - \gamma_\lambda^2}.$$

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We need to consider not only right-moving (k real, > 0) and left-moving (k real, $k < 0$) modes, but also the damped modes, $\omega < c\gamma_\lambda$, with k imaginary. Far from the sources, only the real k modes will matter, but we need all modes for a complete set of states.

Expand our fields in normal modes, indexed by λ, which includes a type (TE or TM or TEM) as well as indices ("quantum numbers") defining the mode. Each mode λ has two k values,
a "positive" one, $k > 0$ real, or k imaginary, Im $k > 0$, and
a "negative" one, $k < 0$ real, or k imaginary, Im $k < 0$.
For each λ let k_λ be the "positive" value (*i.e.* positive real or imaginary with positive imaginary part.)

For the positive mode part of the fields, we have

$$\vec{E}^+_\lambda(x,y,z) = \left[\vec{E}_\lambda(x,y) + \hat{z}E_{z\,\lambda}(x,y)\right] e^{ik_\lambda z}$$
$$\vec{H}^+_\lambda(x,y,z) = \left[\vec{H}_\lambda(x,y) + \hat{z}H_{z\,\lambda}(x,y)\right] e^{ik_\lambda z}$$

where $\vec{E}_\lambda(x,y)$ and $\vec{H}_\lambda(x,y)$ are purely transverse, and are determined by E_z and H_z as in lecture 6 (Jackson 8.26).

The negative modes are found by $z \leftrightarrow -z$, which involves a parity transformation, under which E_z changes sign but the transverse part doesn't. But the magnetic field is a pseudovector, so under parity it behaves the opposite way, and H_z doesn't change sign but $\vec{H}(x,y)$ does. Thus

$$\vec{E}^-_\lambda(x,y,z) = \left[\vec{E}_\lambda(x,y) - \hat{z}E_{z\,\lambda}(x,y)\right] e^{-ik_\lambda z}$$
$$\vec{H}^-_\lambda(x,y,z) = \left[-\vec{H}_\lambda(x,y) + \hat{z}H_{z\,\lambda}(x,y)\right] e^{-ik_\lambda z}$$

The normal modes are determined by solutions of the Helmholtz equation $\left(\nabla_t^2 + \gamma_\lambda^2\right)\psi = 0$, with $\psi|_\Gamma = 0$ for TM modes or Neumann conditions on Γ for the TE modes. With two more indices, $\lambda_{mn}^{\mathrm{TM}}$ give a complete set of functions on the cross section with $\psi|_\Gamma = 0$, and $\lambda_{mn}^{\mathrm{TE}}$ give a complete set of functions on the cross section with $\left.\frac{\partial\psi}{\partial n}\right|_\Gamma = 0$.

Note that if ψ_λ and ψ_μ are solutions to $\left(\nabla_t^2 + \gamma^2\right)\psi = 0$ with $\gamma = \gamma_\lambda$ and $\gamma = \gamma_\mu$ respectively,

$$\begin{aligned}
\int_A \left(\vec{\nabla}_t\psi_\lambda\right)\cdot\left(\vec{\nabla}_t\psi_\mu\right) &= \int_A \left[\vec{\nabla}_t\cdot\left(\psi_\lambda\vec{\nabla}_t\psi_\mu\right) - \psi_\lambda\nabla^2\psi\right] \\
&= \int_\Gamma \psi_\lambda\frac{\partial\psi_\mu}{\partial n} - \int_A \psi_\lambda\nabla_t^2\psi_\mu \\
&= 0 + \gamma_\lambda^2\int_A \psi_\lambda\psi_\mu
\end{aligned}$$

where the vanishing of the $\int_\Gamma$ holds if ψ_λ satisfies Dirichlet boundary conditions $\psi_\lambda|_\Gamma = 0$ or ψ_μ satisfies Neumann conditions $\partial\psi_\mu/\partial n|_\Gamma = 0$.

Reversing $\mu \leftrightarrow \lambda$, we see that if both satisfy the same condition,

$$\int_A \left(\vec{\nabla}_t \psi_\lambda\right) \cdot \left(\vec{\nabla}_t \psi_\mu\right) = \gamma_\lambda^2 \int_A \psi_\lambda \psi_\mu = \gamma_\mu^2 \int_A \psi_\lambda \psi_\mu,$$

so if $\gamma_\lambda \neq \gamma_\mu$, $\int_A \psi_\lambda \psi_\mu = 0$. If there are several solutions with the same γ, with γ^2 real, we may choose them all to be real (or have the same phase), in which case $\int_A \psi_\lambda \psi_\mu$ is a real symmetric matrix which can be diagonalized, so we may *choose* basis functions ψ_λ to be orthonormal under integration over A.

It is convenient to chose the basis functions so that the transverse electric field (at $z = 0$) is real and normalized. For TM modes, $\vec{E} = i\left(k_\lambda/\gamma_\lambda^2\right)\vec{\nabla}_t\psi$, $E_z = \psi$, so we chose ψ to be imaginary for k_λ real and real for k_λ imaginary, with

$$\int_A \psi_\lambda\psi_\mu = -\frac{\gamma_\lambda^2}{k_\lambda^2}\delta_{\lambda\mu}.$$

Then

$$\begin{aligned}\int_A \vec{E}_\lambda\cdot\vec{E}_\mu &= \frac{k_\lambda k_\mu}{\gamma_\lambda^2\gamma_\mu^2}\int_A\left(\vec{\nabla}_t\psi_\lambda\right)\cdot\left(\vec{\nabla}_t\psi_\mu\right)\\ &= \frac{k_\lambda k_\mu}{\gamma_\lambda^2\gamma_\mu^2}\left(\gamma_\lambda^2\int_A\psi_\lambda\psi_\mu\right)\\ &= \frac{k_\lambda k_\mu}{\gamma_\mu^2}\left(\frac{-\gamma_\lambda^2}{k_\lambda^2}\right)\delta_{\lambda\mu} = \delta_{\lambda\mu}.\end{aligned}$$

For TE modes, $\vec{H}_\lambda = i\left(k_\lambda/\gamma_\lambda^2\right)\vec{\nabla}_t\psi_\lambda = \frac{1}{Z_\lambda}\hat{z}\times\vec{E}$, where $Z_\lambda = Z_0 k/k_\lambda$, $k := \omega/c$. Here we choose ψ imaginary (as $k_\lambda Z_\lambda$ is real) with

$$\int_A \psi_\lambda\psi_\mu = \int_A H_{z\lambda}H_{z\mu} = -\frac{\gamma_\lambda^2}{k_\lambda^2 Z_\lambda^2}\delta_{\lambda\mu}.$$

Then for two TE modes

$$\begin{aligned}
\int_A \vec{E}_\lambda\cdot\vec{E}_\mu &= -\frac{k_\lambda k_\mu Z_\lambda Z_\mu}{\gamma_\lambda^2\gamma_\mu^2}\int_A\left(\vec{\nabla}_t\psi_\lambda\right)\cdot\left(\vec{\nabla}_t\psi_\mu\right)\\
&= \frac{k_\lambda k_\mu}{\gamma_\lambda^2\gamma_\mu^2}\left(\int_\Gamma \psi_\lambda\frac{\partial\psi_\mu}{\partial n} - \int_A \psi_\lambda\nabla_t^2\psi_\mu\right)\\
&= \frac{k_\lambda k_\mu}{\gamma_\lambda^2\gamma_\mu^2}\left(0 + \gamma_\lambda^2\int_A\psi_\lambda\psi_\mu\right)\\
&= \frac{k_\lambda k_\mu}{\gamma_\mu^2}\frac{-\gamma_\lambda^2}{k_\lambda^2}\delta_{\lambda\mu} = \delta_{\lambda\mu}.
\end{aligned}$$

Finally, suppose λ is a TM mode with $\vec{E}_\lambda = i\left(k_\lambda/\gamma_\lambda^2\right)\vec{\nabla}_t\psi_\lambda$, and μ is a TE mode, with $\vec{E}_\mu = -i\left(k_\mu Z_\mu/\gamma_\mu^2\right)\hat{z}\times\vec{\nabla}_t\psi_\mu$. Then

$$\begin{aligned}\int_A \vec{E}_\lambda\cdot\vec{E}_\mu &= \frac{k_\lambda k_\mu Z_\mu}{\gamma_\lambda^2\gamma_\mu^2}\int_A\left(\vec{\nabla}_t\psi_\lambda\right)\cdot\left(\hat{z}\times\vec{\nabla}_t\psi_\mu\right)\\ &= -\frac{k_\lambda k_\mu Z_\mu}{\gamma_\lambda^2\gamma_\mu^2}\hat{z}\cdot\int_A\left(\vec{\nabla}_t\psi_\lambda\right)\times\left(\vec{\nabla}_t\psi_\mu\right)\end{aligned}$$

The integral

$$\begin{aligned}\int_A\left(\vec{\nabla}_t\psi_\lambda\right)\times\left(\vec{\nabla}_t\psi_\mu\right) &= \int_A\vec{\nabla}_t\times\left(\psi_\lambda\vec{\nabla}_t\psi_\mu\right)\\ &= \int_\Gamma\psi_\lambda\left(\vec{\nabla}_t\psi_\mu\right)\cdot d\ell = 0\end{aligned}$$

by Stokes theorem and the fact that ψ_λ vanishes on the boundary.

Thus we have shown (or chosen) that the transverse electric fields for the normal modes satisfy

$$\int_A \vec{E}_\lambda \cdot \vec{E}_\mu = \delta_{\lambda\mu}$$

for all the modes, TE and TM.

As we have shown $\displaystyle\int_A \vec{E}_\lambda \cdot \vec{E}_\mu = \delta_{\lambda\mu}$, and as $\vec{H}_\lambda = Z_\lambda^{-1} \hat{z} \times \vec{E}_\lambda$, we have $\displaystyle\int_A \vec{H}_\lambda \cdot \vec{H}_\mu = \frac{1}{Z_\lambda^2} \delta_{\lambda\mu}$, and in calculating the time average power flow $\langle P \rangle = \frac{1}{2} \int_A \left(\vec{E} \times \vec{H} \right) \cdot \hat{z}$ to the right, we can use

$$\int_A \left(\vec{E}_\lambda \times \vec{H}_\mu \right) \cdot \hat{z} = \frac{1}{Z_\lambda} \delta_{\lambda\mu}$$

For a rectangular wave guide

If the cross section is $(0 \leq x \leq a) \times (0 \leq y \leq b)$, the equation separates in x and y, modes are labelled by integers m and n, the number of half wavelengths in each direction, and

TM waves: $\quad \psi|_S = 0$

$$\begin{aligned}
E_{z\,mn} &= \psi = \frac{-2i\gamma_{mn}}{k_\lambda\sqrt{ab}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \\
E_{x\,mn} &= \frac{2\pi m}{\gamma_{mn} a\sqrt{ab}} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \\
E_{y\,mn} &= \frac{2\pi n}{\gamma_{mn} b\sqrt{ab}} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right),
\end{aligned}$$

TE waves: $\left.\frac{\partial\psi}{\partial n}\right|_S = 0$

$$\begin{aligned}
H_{z\,mn} &= \psi = \frac{-2i\gamma_{mn}}{k_\lambda Z_\lambda \sqrt{ab}} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right), \\
E_{x\,mn} &= \frac{-2\pi n}{\gamma_{mn} b\sqrt{ab}} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \\
E_{y\,mn} &= \frac{2\pi m}{\gamma_{mn} a\sqrt{ab}} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right),
\end{aligned}$$

where

$$\gamma_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right).$$

The overall constants are determined from the normalization $\int_A E_x^2 + E_y^2 = 1$, except that for TE modes, we need an extra factor of $1/\sqrt{2}$ for each n or m which is zero, as $\int \cos^2(m\pi x/a) = a(1 + \delta_{m0})/2$.

Expansion of Free Waves

Except where there are sources, the fields are an expansion in terms of normal modes, divided into positive and negative components:

$$\vec{E} = \vec{E}^{+} + \vec{E}^{-}, \qquad \vec{H} = \vec{H}^{+} + \vec{H}^{-},$$

with

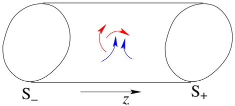
$$E^{\pm} = \sum_{\lambda} A_{\lambda}^{\pm} \vec{E}_{\lambda}^{\pm}, \qquad H^{\pm} = \sum_{\lambda} A_{\lambda}^{\pm} \vec{H}_{\lambda}^{\pm},$$

Coefficients $A_{\lambda}^{\pm}$ are uniquely determined by transverse $\vec{E}$ and $\vec{H}$ along any cross section. For example, at $z = 0$ $\vec{E}$ has expansion coefficients $A_{\lambda}^{+} + A_{\lambda}^{-}$ while $\vec{H}$ has coefficients $A_{\lambda}^{+} - A_{\lambda}^{-}$. From the orthonormality properties we find

$$A_{\lambda}^{\pm} = \frac{1}{2} \int_{A} \vec{E} \cdot \vec{E}_{\lambda} \pm Z_{\lambda}^{2} \vec{H} \cdot \vec{H}_{\lambda}.$$

Localized Sources

Now consider a source $\vec{J}(\vec{x})e^{-i\omega t}$ confined to some region $z \in [z_-, z_+]$. Consider cross sections S_- and S_+, with all sources between them.



so at S_+ there is no amplitude for any mode with negative k or with $-i|k|$, which would represent left-moving waves or exponential blow up (as $z \to +\infty$). The reverse is true at S_-, so

$$\vec{E} = \sum_{\lambda'} A^+_{\lambda'} \vec{E}^+_{\lambda'}, \qquad \vec{H} = \sum_{\lambda'} A^+_{\lambda'} \vec{H}^+_{\lambda'} \qquad \text{at } S_+$$

$$\vec{E} = \sum_{\lambda'} A^-_{\lambda'} \vec{E}^-_{\lambda'}, \qquad \vec{H} = \sum_{\lambda'} A^-_{\lambda'} \vec{H}^-_{\lambda'} \qquad \text{at } S_-$$

In between, we have the full Maxwell equations (with sources),

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = i\omega\mu_0\vec{H}, \quad \vec{\nabla}\times\vec{H} = \vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t} = \vec{J}-i\omega\epsilon_0\vec{E},$$

while the normal modes obey Maxwell equations without sources:

$$\vec{\nabla}\times\vec{H}_\lambda^\pm = -i\omega\epsilon_0\vec{E}_\lambda^\pm, \quad \vec{\nabla}\times\vec{E}_\lambda^\pm = i\omega\mu_0\vec{H}_\lambda^\pm.$$

If we apply the identity
$\vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right) = \left(\vec{\nabla}\times\vec{A}\right)\cdot\vec{B} - \vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)$, we find

$$\begin{aligned}
&\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}_\lambda^\pm - \vec{E}_\lambda^\pm\times\vec{H}\right) \\
&\quad = \left(\vec{\nabla}\times\vec{E}\right)\cdot\vec{H}_\lambda^\pm - \vec{E}\cdot\left(\vec{\nabla}\times\vec{H}_\lambda^\pm\right) \\
&\qquad - \left(\vec{\nabla}\times\vec{E}_\lambda^\pm\right)\cdot\vec{H} + \vec{E}_\lambda^\pm\cdot\left(\vec{\nabla}\times\vec{H}\right) \\
&\quad = i\omega\mu_0\vec{H}\cdot\vec{H}_\lambda^\pm + i\omega\epsilon_0\vec{E}\cdot\vec{E}_\lambda^\pm - i\omega\mu_0\vec{H}_\lambda^\pm\cdot\vec{H} \\
&\qquad + \vec{E}_\lambda^\pm\cdot\left(\vec{J}-i\omega\epsilon_0\vec{E}\right) \quad = \quad \vec{J}\cdot\vec{E}_\lambda^\pm
\end{aligned}$$

If we integrate this over the volume between S_- and S_+, using Gauss' theorem and the boundary condition that $\vec{E}_\parallel = 0$ at the conductor's surface,

$$\int_S \left(\vec{E} \times \vec{H}^\pm_\lambda - \vec{E}^\pm_\lambda \times \vec{H}\right) \cdot \hat{n} = \int_V \vec{J} \cdot \vec{E}^\pm_\lambda,$$

where S consists of S_+ with $\hat{n} = \hat{z}$, and S_- with $\hat{n} = -\hat{z}$. Let's take the upper sign. The contribution from S_+ is can be reduced to an integral over A at $z = 0$:

$$\begin{aligned}
&\sum_{\lambda'} A^+_{\lambda'} \int_{S_+} \left(\vec{E}^+_{\lambda'} \times \vec{H}^+_\lambda - \vec{E}^+_\lambda \times \vec{H}^+_{\lambda'}\right)_z \\
&= \sum_{\lambda'} A^+_{\lambda'} \int_{S_+} \left(\vec{E}_{\lambda'} \times \vec{H}_\lambda - \vec{E}_\lambda \times \vec{H}_{\lambda'}\right)_z e^{i(k_\lambda + k_{\lambda'})z} \\
&= \sum_{\lambda'} A^+_{\lambda'} \int_A \Big(\vec{E}_{\lambda'} \times \left(Z^{-1}_\lambda \hat{z} \times \vec{E}_\lambda\right) \\
&\qquad\qquad - \vec{E}_\lambda \times \left(Z^{-1}_{\lambda'} \hat{z} \times \vec{E}_{\lambda'}\right)\Big)_z e^{i(k_\lambda + k_{\lambda'})z}
\end{aligned}$$

$$\sum_{\lambda'} A^+_{\lambda'} \int_{S_+} \left(\vec{E}^+_{\lambda'} \times \vec{H}^+_{\lambda} - \vec{E}^+_{\lambda} \times \vec{H}^+_{\lambda'}\right)_z$$

$$= \sum_{\lambda'} A^+_{\lambda'} \int_A \left(\frac{1}{Z_\lambda}\vec{E}_{\lambda'} \cdot \vec{E}_\lambda - \frac{1}{Z_{\lambda'}}\vec{E}_\lambda \cdot \vec{E}_{\lambda'}\right) e^{i(k_\lambda + k_{\lambda'})z}$$

$$= \sum_{\lambda'} A^+_{\lambda'} \delta_{\lambda\lambda'} \left(\frac{1}{Z_\lambda} - \frac{1}{Z'_\lambda}\right) e^{i(k_\lambda + k_{\lambda'})z} = 0.$$

On the other hand, the contribution from S_- is

$$\sum_{\lambda'} A^-_{\lambda'} \int_{S_-} - \left(\vec{E}^-_{\lambda'} \times \vec{H}^+_{\lambda} - \vec{E}^+_{\lambda} \times \vec{H}^-_{\lambda'}\right) \cdot \hat{z}$$

$$= \sum_{\lambda'} A^-_{\lambda'} \int_{S_-} -\left(\vec{E}_{\lambda'} \times \vec{H}_{\lambda} + \vec{E}_{\lambda} \times \vec{H}_{\lambda'}\right) \cdot \hat{z}\, e^{i(k_\lambda - k_{\lambda'})z}$$

$$= -\sum_{\lambda'} A^-_{\lambda'} \frac{2}{Z_\lambda} \delta_{\lambda\lambda'} = -\frac{2}{Z_\lambda} A^-_\lambda$$

so $$A^-_\lambda = -\frac{Z_\lambda}{2} \int_V \vec{J} \cdot \vec{E}^+_\lambda.$$

The same argument for the lower sign, as spelled out in the book, gives the equation with the superscript signs reversed, so both are

$$A_\lambda^{\pm} = -\frac{Z_\lambda}{2}\int_V \vec{J}\cdot\vec{E}_\lambda^{\mp}.$$

In addition to sources due to currents, we may have contributions due to obstacles or holes in the conducting boundaries. These can be treated as additional surface terms in Gauss' law (by excluding obstacles from the region of integration V), but this requires knowing the full fields at the surface of the obstacles or the missing parts of the waveguide conductor. This is treated in §9.5B, but we won't discuss it here.
So finally we are at the end of Chapter 8.