Schumann Resonances

Resonant cavities do not need to be cylindrical, of course. The surface of the Earth ($R_E \approx 6400$ km) and the ionosphere ($R = R_E + h$, $h \approx 100$ km) form concentric spheres which are sufficiently good conductors to form a resonant cavity.
Take $\vec{E}, \vec{H} \propto e^{-i\omega t}$, cavity essentially vacuum.
$Z_0 = \sqrt{\mu_0/\epsilon_0}, \quad c = 1/\sqrt{\mu_0\epsilon_0}$. Set $k = \omega/c$.

Maxwell:

$$\vec{\nabla}\times\vec{E} = ikZ_0\vec{H}, \quad \vec{\nabla}\times\vec{H} = -i\frac{k}{Z_0}\vec{E}, \quad \vec{\nabla}\cdot\vec{E} = 0, \quad \vec{\nabla}\cdot\vec{H} = 0,$$

So

$$\begin{aligned}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{H}\right) &= \vec{\nabla}\times\left(-i\frac{k}{Z_0}\vec{E}\right) = k^2\vec{H} \\ &= \vec{\nabla}\underbrace{\left(\vec{\nabla}\cdot\vec{H}\right)}_{0} - \nabla^2\vec{H}\end{aligned}$$

So

$$\left(\nabla^2 + k^2\right)\vec{H} = 0, \quad \vec{\nabla}\cdot\vec{H} = 0, \quad \text{and } \vec{E} = i\frac{Z_0}{k}\vec{\nabla}\times\vec{H}.$$

Similiarly we can derive

$$\left(\nabla^2 + k^2\right)\vec{E} = 0, \quad \vec{\nabla}\cdot\vec{E} = 0, \quad \text{and } \vec{H} = -i\frac{1}{kZ_0}\vec{\nabla}\times\vec{E}.$$

Each cartesian component obeys Helmholtz, but the radial component $\vec{r}\cdot\vec{A}$ (for $\vec{A}$ either $\vec{E}$ or $\vec{H}$) is more suitable to look at.

$$\begin{aligned}\nabla^2(\vec{r}\cdot\vec{A}) &= \sum_{ij}\frac{\partial^2}{\partial r_i^2}(r_jA_j) = \sum_{ij}\left(r_j\frac{\partial^2}{\partial r_i^2}A_j + 2\frac{\partial A_j}{\partial r_i}\delta_{ij}\right) \\ &= \vec{r}\cdot\nabla^2\vec{A} + 2\underbrace{\vec{\nabla}\cdot\vec{A}}_{=0 \text{ for } \vec{E},\vec{H}}.\end{aligned}$$

$$\text{so} \qquad \left(\nabla^2 + k^2\right)(\vec{r}\cdot\vec{E}) = 0, \qquad \left(\nabla^2 + k^2\right)(\vec{r}\cdot\vec{H}) = 0.$$

Magnetic multipole field: $\vec{r}\cdot\vec{E} \equiv 0$,
Electric multipole field: $\vec{r}\cdot\vec{H} \equiv 0$.
Whichever isn't identically zero satisfies Helmholtz.

Separation of variables

In spherical coordinates,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].$$

Thus solutions of Helmholtz's equation are found by separation of variables, $F(r)Y(\theta,\phi)$, where the angular part satisfies

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\theta^2}\right]Y_{\ell m} = \ell(\ell+1)Y_{\ell m}.$$

This you should recognize from Quantum Mechanics as the equation for the spherical harmonics.
Single-valuedness for corresponding values of θ and ϕ require $\ell \in \mathbb{Z}$.

Thus the solutions are

$$\begin{aligned}&\text{TE:} \qquad \vec{r}\cdot\vec{H}^{(M)}_{\ell m} = \frac{\ell(\ell+1)}{k}g_\ell(kr)Y_{\ell m}(\theta,\phi), \quad \vec{r}\cdot\vec{E}^{(M)} = 0 \\ &\text{TM:} \qquad \vec{r}\cdot\vec{E}^{(E)}_{\ell m} = -Z_0\frac{\ell(\ell+1)}{k}f_\ell(kr)Y_{\ell m}(\theta,\phi), \; \vec{r}\cdot\vec{H}^{(E)} = 0.\end{aligned}$$

In fact, let's steal more from quantum mechanics. Define the operators $\vec{L} = -i\vec{r}\times\vec{\nabla}$.

$$L_\pm = L_x \pm iL_y = e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right), \quad L_z = -i\frac{\partial}{\partial\phi},$$

and we recall

$$L_\pm Y_{\ell m} = \sqrt{(\ell\mp m)(\ell\pm m+1)}\,Y_{\ell,m\pm1}, \quad L_zY_{\ell m} = mY_{\ell m},$$

$$L^2Y_{\ell m} = \ell(\ell+1)Y_{\ell m}.$$

Dotting $\vec{r}$ into the first Maxwell equation,

$$ikZ_0\vec{r}\cdot\vec{H} = \vec{r}\cdot\left(\vec{\nabla}\times\vec{E}\right) = \left(\vec{r}\times\vec{\nabla}\right)\cdot\vec{E} = i\vec{L}\cdot\vec{E},$$

so for the magnetic multipole (TE) field

$$\vec{L}\cdot\vec{E}^{(M)}_{\ell m} = kZ_0\vec{r}\cdot\vec{H} = Z_0g_\ell(kr)L^2Y_{\ell m},$$

which at least hints at

$$\vec{E}^{(M)}_{\ell m} = Z_0g_\ell(kr)\vec{L}Y_{\ell m}. \qquad (1)$$

Also, this is consistent with $\vec{r}\cdot\vec{E}^{(M)}_{\ell m} = 0$ as $\vec{r}\cdot\vec{L} = -i\vec{r}\cdot\left(\vec{r}\times\vec{\nabla}\right) = 0$.

The rest of the fields in a magnetic multipole are

$$\vec{H}^{(M)}_{\ell m} = -\frac{i}{kZ_0}\vec{\nabla}\times\vec{E}^{(M)}_{\ell m}.$$

This magnetic multipole field configuration is also called transverse electric (TE), as $\vec{E}$ is transverse to the radial direction.

The same holds for the electric multipole (TM) field:

$$\begin{aligned}\vec{H}^{(E)}_{\ell m} &= f_\ell(kr)\vec{L}Y_{\ell m}(\theta,\phi), \\ \vec{E}^{(E)}_{\ell m} &= i\frac{Z_0}{k}\vec{\nabla}\times\vec{H}^{(E)}_{\ell m} = \frac{Z_0}{k}\vec{\nabla}\times\left(\vec{r}\times\vec{\nabla}\right)f_\ell(kr)Y_{\ell m}(\theta,\phi).\end{aligned}$$

But $\vec{\nabla}\times\left(\vec{r}\times\vec{\nabla}\right) = \vec{r}\nabla^2 - \vec{\nabla}\left(1 + r\frac{\partial}{\partial r}\right)$, so

$$\vec{E}^{(E)}_{\ell m} = \frac{Z_0}{k}\left[\vec{r}\nabla^2 - \vec{\nabla}\left(1 + r\frac{\partial}{\partial r}\right)\right]f_\ell(kr)Y_{\ell m}(\theta,\phi).$$

the transverse part of the electric field is determined by

$$\begin{aligned}
\vec r \times E^{(E)}_{\ell m} &= \frac{Z_0}{k}\,\vec r \times \left(\vec r\,\nabla^2 - \vec\nabla\left(1 + r\frac{\partial}{\partial r}\right)\right) f_\ell(kr) Y_{\ell m}(\theta,\phi)\\
&= -\frac{Z_0}{k}\left(\vec r \times \vec\nabla\right)\left(1 + r\frac{\partial}{\partial r}\right) f_\ell(kr) Y_{\ell m}(\theta,\phi)\\
&= -i\frac{Z_0}{k}\left[\left(1 + r\frac{\partial}{\partial r}\right) f_\ell(kr)\right]\left[\vec L\, Y_{\ell m}(\theta,\phi)\right].
\end{aligned}$$

Now we need $\vec r \times E^{(E)}_{\ell m} = 0$ at $r = R_E$ and $r = R_E + h$. Note for $\ell = 0$ we have spherical symmetry, $vecE$ and $\vec H$ are purely radial and angle-independent, so then $\vec\nabla \cdot \vec E = 0 \Longrightarrow \vec E \equiv c/r^2$, and we have a solution only for $k = 0$ and this is a static coulomb field. For $\ell \neq 0$, vanishing requires $\left(1 + r\frac{\partial}{\partial r}\right) f_\ell(kr) = 0$ at $r = R_E$ and $r = R_E + h$. If, instead, we look for a magnetic multipole solution, we need $g_\ell(kr) = 0$ at $r = R_E$ and $r = R_E + h$.

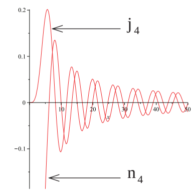
Solution of Radial Equation

As $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial}{\partial r} - \dfrac{1}{r^2}L^2$, the radial part of an (ℓ, m) mode satisfies

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} + k^2\right) g_\ell(kr) = 0.$$

The same equation holds for $f_\ell(kr)$.

Solutions are *spherical* Bessel and Hankel functions, similar to $\sin(kr)$ and $\cos(kr)$. Easy to make combinations which vanish at two points h apart, with k of order π/h. For $h \sim 100$ km, frequency ~ 10 kHz. Radio waves are higher frequency, and we could use geometrical optics to describe what happens.



$j_4(x)$ and $n_4(x)$

We could have resonant magnetic multipole (TE) fields in the kilohertz range. But there are observed resonances at 8, 14, and 20 Hertz! Why? We need to look at solutions more closely.

Our equation is

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2}\right) f_\ell(r) = 0,$$

This can be transformed in several useful ways. Fiddle the scale of r and the multiply by a power of r,

$$\begin{aligned}
f_\ell(r) &= \frac{u_{\ell,\alpha,\beta}(\beta k r)}{(\beta k r)^\alpha}\\
\Longrightarrow &\left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + \frac{1}{\beta^2} - \frac{\ell(\ell+1)}{x^2}\right)\frac{u_{\ell,\alpha,\beta}(x)}{x^\alpha} = 0
\end{aligned}$$

$$\left(\frac{d^2}{dx^2} + \frac{2(1-\alpha)}{x}\frac{d}{dx} + \frac{1}{\beta^2} + \frac{\alpha(\alpha-1) - \ell(\ell+1)}{x^2}\right) u_{\ell,\alpha,\beta}(x) = 0.$$

Two useful choices for α and β:
Choice 1: $\alpha = 1/2, \beta = 1$

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{(\ell + 1/2)^2}{x^2}\right) u_{\ell,\frac12,1}(x) = 0,$$

This is Bessel's equation with $\nu = \ell + \frac12$, solutions $u = aJ_{\ell+\frac12}(kr) + bN_{\ell+\frac12}(kr)$, and $f_\ell(r) = a' j_\ell(kr) + b' n_\ell(kr)$, where j and n are *spherical Bessel* and *spherical Neumann* functions:

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell+1/2}(x), \qquad n_\ell(x) = \sqrt{\frac{\pi}{2x}} N_{\ell+1/2}(x)$$

$$h_\ell^{(1,2)}(x) = \sqrt{\frac{\pi}{2x}}\left(J_{\ell+1/2}(x) \pm i\, N_{\ell+1/2}(x)\right).$$

Choice 2: $\alpha = 1$, $\beta = 1/\sqrt{\ell(\ell+1)}$,

$$\left(\frac{d^2}{dx^2} + \ell(\ell+1)\left(1 - \frac{1}{x^2}\right)\right) u_\ell = 0, \qquad (2)$$

As $f \propto u/r$, the boundary conditions for an electric multipole (TM) field at $x = \beta k R_E$ and $x = \beta k(R_E + h)$ are

$$\left(1 + r\frac{d}{dr}\right)\frac{u(\beta k r)}{\beta k r} = 0 = du/dx, \quad \text{with} \quad x = \beta k r.$$

To get du/dx to vanish at nearby x's is now easy. Of course the average value of d^2u/dx^2 has to be zero between the two zeroes of du/dx, but that is assured by (2) for $x = 1$ roughly in the center of the interval, so $1 \approx \beta k R_E$, or

$$k \approx \frac{\sqrt{\ell(\ell+1)}}{R_E}, \qquad f = \frac{c}{2\pi}\frac{\sqrt{\ell(\ell+1)}}{R_E} = 7.46\sqrt{\ell(\ell+1)}$$

Hz = 10.5 Hz, 18.3 Hz, 25.8 Hz, ... The observed resonant frequecies are about 20% lower, said to be due to imperfect conductivity of the ground and ionosphere.