Wave velocities

Wave guide traveling modes in z direction

$$\vec{E}, \vec{B} \propto e^{ikz - i\omega t}$$

with dispersion relation $k^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2$.
Same form as for high-frequencies in dielectrics (Jackson 7.61), with $\omega_\lambda \sim$ plasma frequency.
Phase velocity $v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} \frac{1}{\sqrt{1 - \left(\frac{\omega_\lambda}{\omega}\right)^2}} > \frac{1}{\sqrt{\mu\epsilon}}$,

greater than for unbounded.

Group velocity $v_g = \frac{d\omega}{dk} = \frac{1}{\mu\epsilon}\frac{k}{\omega} = \frac{1}{\mu\epsilon}\frac{1}{v_p} < \frac{1}{\sqrt{\mu\epsilon}}$, less than for the unbounded medium. (I used $kdk = \mu\epsilon\omega d\omega$)

Attenuation

Last slide assumed perfectly conducting walls. Real walls have energy lose, attenuation, k develops small positive imaginary part $i\beta$ (so extra $e^{-\beta z}$ factor for $\vec{E}$ and for $\vec{H}$).
Find β by comparing power lost per unit length to power transmitted.
Power is quadratic in fields. Only real parts of fields are real.
Poynting vector $\vec{S}_{\text{phys}} = \vec{E}_{\text{phys}} \times \vec{H}_{\text{phys}}$ needs

$$\begin{aligned}\vec{E}_{\text{phys}}&(x, y, z, t) \\ &= \frac{1}{2}\left(\vec{E}(x, y, k, \omega)e^{ikz-i\omega t} + \vec{E}^*(x, y, k, \omega)e^{-ikz+i\omega t}\right).\end{aligned}$$

and similarly for $\vec{H}$.

So

$$
\begin{aligned}
\vec{S}_{\text{phys}} &= \vec{E}_{\text{phys}} \times \vec{H}_{\text{phys}} \\
&= \frac{1}{4}\Bigg(\left(\vec{E}(x,y,k,\omega) e^{ikz-i\omega t} + \vec{E}^*(x,y,k,\omega) e^{-ikz+i\omega t} \right) \times \\
&\quad \left(\vec{H}(x,y,k,\omega) e^{ikz-i\omega t} + \vec{H}^*(x,y,k,\omega) e^{-ikz+i\omega t} \right) \Bigg) \\
&= \frac{1}{4}\Bigg(\vec{E}(x,y,k,\omega) \times \vec{H}(x,y,k,\omega) e^{2ikz-2i\omega t} \\
&\quad + \vec{E}^*(x,y,k,\omega) \times \vec{H}(x,y,k,\omega) \\
&\quad + \vec{E}(x,y,k,\omega) \times \vec{H}^*(x,y,k,\omega) \\
&\quad + \vec{E}^*(x,y,k,\omega) \times \vec{H}^*(x,y,k,\omega) e^{-2ikz+2i\omega t} \Bigg)
\end{aligned}
$$

First and last terms rapidly oscillating, average to zero, so

$$
\begin{aligned}
\langle \vec{S} \rangle &= \frac{1}{4}\Big(\vec{E}^*(x,y,k,\omega) \times \vec{H}(x,y,k,\omega) \\
&\qquad + \vec{E}(x,y,k,\omega) \times \vec{H}^*(x,y,k,\omega)\Big) \\
&= \frac{1}{2}\mathrm{Re}\ \Big(\vec{E}(x,y,k,\omega) \times \vec{H}^*(x,y,k,\omega)\Big)
\end{aligned}
$$

Define the complex $\vec{S} := \frac{1}{2}\left(\vec{E} \times \vec{H}^*\right)$, with the physical average flux given by the real part.
Power flow $\propto \int \hat{z} \cdot \mathrm{Re}\ \vec{S}$, so only the transverse parts of $\vec{E}$ and $\vec{H}$ are needed. Recall

TM: $\quad E_z = \psi, \quad \vec{E}_t = i\frac{k}{\gamma_\lambda^2}\vec{\nabla}_t\psi, \quad \vec{H}_t = i\frac{\epsilon\omega}{\gamma_\lambda^2}\hat{z} \times \vec{\nabla}_t\psi$

TE: $\quad H_z = \psi, \quad \vec{H}_t = i\frac{k}{\gamma_\lambda^2}\vec{\nabla}_t\psi, \quad \vec{E}_t = -i\frac{\mu\omega}{\gamma_\lambda^2}\hat{z} \times \vec{\nabla}_t\psi$

As $\hat{z}\cdot\left(\vec{\nabla}_t\psi \times \hat{z} \times \vec{\nabla}_t\psi^*\right) = \left|\vec{\nabla}_t\psi\right|^2$, we have

$$P = \hat{z}\cdot\int_A \mathrm{Re}\ S = \frac{\omega k}{2\gamma_\lambda^4}\int_A |\vec{\nabla}_t\psi|^2 \cdot \begin{cases} \epsilon & \text{(for TM)} \\ \mu & \text{(for TE)} \end{cases}$$

The integral

$$\int_A |\vec{\nabla}_t\psi|^2 = \oint_S \psi^*\frac{\partial\psi}{\partial n} - \int_A \psi^*\nabla_t^2\psi = 0 + \gamma_\lambda^2\int_A \psi^*\psi.$$

As $\omega_\lambda := \gamma_\lambda/\sqrt{\mu\epsilon}$, $k = \omega\sqrt{\mu\epsilon}\sqrt{1-\omega_\lambda^2/\omega^2}$,

$$P = \frac{1}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}\int_A \psi^*\psi \cdot \begin{cases} \epsilon & \text{(for TM)} \\ \mu & \text{(for TE)} \end{cases}$$

Energy Density

Energy per unit length
$U = \int_A u = \frac{1}{2}\int_A \left(\vec{E}_{\text{phys}} \cdot \vec{D}_{\text{phys}} + \vec{B}_{\text{phys}} \cdot \vec{H}_{\text{phys}}\right)$,

$$\langle U \rangle = \frac{1}{4}\int_A \epsilon|\vec{E}|^2 + \mu|\vec{H}|^2$$

Need z components (ψ or 0) as well as transverse ones. Plugging in is straightforward (see notes), and we find

$$\langle U \rangle = \frac{\omega^2}{2\omega_\lambda^2}\int_A |\psi|^2 \times \begin{cases} \epsilon & \text{TM mode} \\ \mu & \text{TE mode} \end{cases}$$

In either case,

$$\frac{\langle P \rangle}{\langle U \rangle} = \frac{1}{\sqrt{\epsilon\mu}}\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} = v_g.$$

Energy flux = energy density times *group* velocity.

Attenuation and Power Loss

At an interface, we found power loss per unit area is

$$\frac{1}{2\delta\sigma}\left|\vec{H}_{\parallel}\right|^2 = \frac{1}{2\delta\sigma}\left|\hat{n}\times\vec{H}\right|^2,$$

with conductivity σ and skin depth $\delta = \sqrt{2/\mu_c\sigma\omega}$. As the power drops off as the square of the fields, so as $e^{-2\beta z}$

$$\frac{dP}{dz} = -2\beta P(z) = -\frac{1}{2\delta\sigma}\oint_{\Gamma}\left|\hat{n}\times\vec{H}\right|^2 d\ell,$$

where the integral $d\ell$ is over the loop Γ around the interface at fixed z.
β will depend on the mode being considered, so we will call it β_λ.
Note resistivity can couple modes, but we will not discuss that.

Attenuation for TM modes

For a TM mode,

$$\hat{n} \times \vec{H} = \hat{n} \times \vec{H}_t = \frac{i\epsilon\omega}{\gamma_\lambda^2} \hat{n} \times (\hat{z} \times \vec{\nabla}_t \psi) = \frac{i\epsilon\omega}{\gamma_\lambda^2} \left(\hat{n} \cdot \vec{\nabla}_t \psi \right) \hat{z}$$

so

$$\begin{aligned} \beta_\lambda &= \frac{1}{4\sigma\delta} \left(\frac{\epsilon\omega}{\gamma_\lambda^2} \right)^2 \int_\Gamma \left| \frac{\partial \psi}{\partial n} \right|^2 \Big/ \frac{\omega k \epsilon}{2\gamma_\lambda^4} \int_A \left| \vec{\nabla} \psi \right|^2 \\ &= \frac{\omega\epsilon}{2k\sigma\delta} \underbrace{\int_\Gamma \left| \frac{\partial \psi}{\partial n} \right|^2 \Big/ \int_A \left| \vec{\nabla} \psi \right|^2}_{C\xi_\lambda / A} \end{aligned}$$

where C is the length of Γ and A the area, and ξ_λ is a mode– and geometry–dependent dimensionless number, the average size of the normal derivative to the gradient, which we would expect to be of order 1.

Attenuation for TE modes

For a TE mode, $\hat{n} \times \vec{H} = \hat{n} \times \vec{H}_t + \hat{n} \times \hat{z} H_z$ so

$$\left|\hat{n} \times \vec{H}\right|^2 = \left|\hat{n} \times \vec{H}_t\right|^2 + |H_z|^2 = \left(\frac{k}{\gamma_\lambda^2}\right)^2 \left|\hat{n} \times \vec{\nabla}_t \psi\right|^2 + |\psi|^2.$$

Again let us write

$$\int_\Gamma \left|\hat{n} \times \vec{\nabla}_t \psi\right|^2 \Big/ \int_A \left|\vec{\nabla} \psi\right|^2 = \frac{C}{A} \xi_\lambda, \quad \int_\Gamma |\psi|^2 \Big/ \int_A |\psi|^2 = \frac{C}{A} \zeta_\lambda.$$

where ζ_λ is another dimensionless number of order one, and ξ_λ is somewhat differently defined. Then

$$\int_\Gamma \left|\hat{n} \times \vec{\nabla}_t \psi\right|^2 \Big/ \int_A |\psi|^2 = \gamma_\lambda^2 \frac{C}{A} \xi_\lambda.$$

Frequency Dependence

The conductivity, permeability and permittivity may be considered approximately frequency-independent, but the skin depth δ goes as $\omega^{-1/2}$, so let us write $\delta = \delta_\lambda \sqrt{\omega_\lambda/\omega}$. Then we can extract the frequency dependence of the attenuation factors

TM mode:

$$\beta_\lambda = \sqrt{\frac{\epsilon}{\mu}} \frac{1}{\sigma\delta_\lambda} \frac{C}{2A} \frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}}} \xi_\lambda.$$

TE mode:

$$\beta_\lambda = \sqrt{\frac{\epsilon}{\mu}} \frac{1}{\sigma\delta_\lambda} \frac{C}{2A} \frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}}} \left[\xi_\lambda + \eta_\lambda \left(\frac{\omega_\lambda}{\omega}\right)^2\right],$$

where $\eta_\lambda = \zeta_\lambda - \xi_\lambda$.

Note that β_λ diverges as we approach the cutoff frequency $\omega \to \omega_\lambda$,
and $\beta_\lambda \sim \sqrt{\omega}$ as $\omega \to \infty$.
Thus there is a minimum, at $\sqrt{3}\,\omega_\lambda$ for TM, and at a geometry-dependent value for TE modes.

We will skip section 8.6

Attenuation in a circular wave guide

We found the modes for a circular wave guide of radius r are given by

$$\psi_{mn}^{\mathrm{TE}} = J_m(x'_{mn}\rho/r)\cos m\phi, \qquad \text{with} \qquad \frac{dJ_m}{dx}(x'_{mn}) = 0,$$
$$\psi_{mn}^{\mathrm{TM}} = J_m(x_{mn}\rho/r)\cos m\phi, \qquad \text{with} \qquad J_m(x'_{mn}) = 0$$

The cutoff wavenumbers and frequencies are $\gamma_{mn}^{\mathrm{TE}} = x'_{mn}/r$ and $\gamma_{mn}^{\mathrm{TM}} = x_{mn}/r$, with $\omega_\lambda = c\gamma_\lambda$.
We also found for general cylindrical wave guides that the attenuation coefficients are

TM mode: $$\beta_\lambda = \sqrt{\frac{\epsilon}{\mu}}\frac{1}{\sigma\delta_\lambda}\frac{C}{2A}\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}}\xi_\lambda.$$

TE mode: $$\beta_\lambda = \sqrt{\frac{\epsilon}{\mu}}\frac{1}{\sigma\delta_\lambda}\frac{C}{2A}\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}}\left[\xi_\lambda + \eta_\lambda\left(\frac{\omega_\lambda}{\omega}\right)^2\right],$$

The dimensionless quantities ξ_λ, ζ_λ and η_λ are given by

$$\frac{C}{A}\xi_\lambda^{\rm TM} = \int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \Bigg/ \int_A \left|\vec\nabla\psi\right|^2,$$

$$\begin{aligned}\frac{C}{A}\xi_\lambda^{\rm TE} &= \int_\Gamma \left|\hat n\times\vec\nabla_t\psi\right|^2 \Big/ \gamma_\lambda^2\int_A |\psi|^2,\\ \frac{C}{A}\zeta_\lambda^{\rm TE} &= \int_\Gamma |\psi|^2 \Big/ \int_A |\psi|^2,\end{aligned}$$

and $\eta_\lambda^{\rm TE} = \zeta_\lambda^{\rm TE} - \xi_\lambda^{\rm TE}$.

As $\psi(\rho,\phi) = J_m(\gamma\rho)\cos m\phi$.

$$\frac{\partial\psi}{\partial n} = \gamma J_m'(\gamma r)\cos m\phi, \qquad \hat n\times\vec\nabla_t\psi = \frac{m}{\rho}J_m(\rho)\sin m\phi.$$

The angular integrals are in all case trivial (and even more so if we used the complex modes $e^{-m\phi}$).

For TM:

$$\int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 = r\int_0^{2\pi} d\phi\gamma^2 J_m'^2(\gamma r)\cos^2\phi = \pi r\gamma^2 J_m'^2(\gamma r)(1+\delta_{m0}),$$

For TE:

$$\int_\Gamma |\psi|^2 = rJ_m^2(x'_{mn})\int_0^{2\pi}\cos^2 m\phi\, d\phi = \pi r J_m^2(x'_{mn})(1+\delta_{m0}),$$

$$\begin{aligned}\int_\Gamma |\hat{n}\times\nabla_t\psi|^2 &= r\int_0^{2\pi} d\phi\left(\frac{\partial\psi}{r\partial\phi}\right)^2 \\ &= \frac{1}{r}J_m^2(x'_{mn})\int_0^{2\pi}(m\sin m\phi)^2 \\ &= \frac{\pi m^2}{r}J_m^2(x'_{mn})(1+\delta_{m0}),\end{aligned}$$

For both modes, we need the nontrivial

$$\begin{aligned}\int_A \psi^2 &= \int_0^r \rho d\rho J_m^2(\gamma\rho)\int_0^{2\pi} d\phi\cos^2(m\phi) \\ &= \pi(1+\delta_{m0})\int_0^r \rho d\rho J_m^2(\gamma\rho)\end{aligned}$$

The radial integral

$$\int_0^r \rho d\rho J_m^2(\gamma\rho) = r^2 \int_0^1 u du J_m^2(xu),$$

where x is either x_{mn} (for TM) or x'_{mn} (for TE). The integral is related to the orthonormalization properties of Bessel functions. From Arfken (or "Lecture Notes" $\rightarrow$ "Notes on Bessel functions") we find

$$\int_0^1 \left[J_m\left(x_{mn}u\right)\right]^2 u du = \frac{1}{2} J_{m+1}^2(x_{mn})$$

$$\int_0^1 \left[J_m\left(x'_{mn}u\right)\right]^2 u du = \frac{1}{2}\left(1 - \frac{m^2}{(x'_{mn})^2}\right) J_m^2(x'_{mn})$$

Thus for the TM modes, we have

$$
\begin{aligned}
\frac{C}{A}\xi_{mn}^{\mathrm{TM}} &= \int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \bigg/ (\gamma_{mn}^{\mathrm{TM}})^2 \int_A \psi^2 = \frac{\pi r J_m'^{\,2}(x_{mn})}{\frac{\pi r^2}{2} J_{m+1}^2(x_{mn})} \\
&= \frac{2}{r}\frac{J_m'^{\,2}(x_{mn})}{J_{m+1}^2(x_{mn})}
\end{aligned}
$$

In fact, there is an identity (see footnote again) $J_m'(x) = \frac{m}{x}J_m(x) - J_{m+1}(x)$, which means, as $J_m(x_{mn}) = 0$, that $J_m'(x_{mn}) = -J_{m+1}(x_{mn})$, $\dfrac{C}{A}\xi_{mn}^{\mathrm{TM}} = \dfrac{2}{r}$, and

$$
\beta_{mn}^{\mathrm{TM}} = \sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{r\sigma\delta_\lambda}\,\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\dfrac{\omega_\lambda^2}{\omega^2}}}
$$

for all TM modes.

For the TE modes,

$$\begin{aligned}
\frac{C}{A}\xi^{\mathrm{TE}}_{mn} &= \int_\Gamma |\hat{n}\times\nabla_t\psi|^2 \Big/ (\gamma^{\mathrm{TE}}_{mn})^2 \int_A \psi^2 \\
&= \frac{m^2\pi J_m^2(x'_{mn})/r}{\pi(\gamma^{\mathrm{TE}}_{mn})^2 r^2 \frac{1}{2}\left(1-(m/x'_{mn})^2\right) J_m^2(x'_{mn})} \\
&= \frac{2m^2}{r(x'^2_{mn}-m^2)}.
\end{aligned}$$

$$\begin{aligned}
\frac{C}{A}\zeta^{\mathrm{TE}}_{mn} &= \int_\Gamma |\psi|^2 \Big/ \int_A \psi^2 = \frac{\pi r J_m^2(x'_{mn})}{\frac{\pi}{2}\left(1-(m/(x'_{mn})^2)\right) J_m^2(x'_{mn})} \\
&= \frac{2x'^2_{mn}}{r\,(x'^2_{mn}-m^2)}.
\end{aligned}$$

So the attenuation coefficient is

$$\beta^{\mathrm{TE}}_{mn} = \sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{r\sigma\delta_\lambda}\,\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}}\left[\frac{1}{(x'^2_{mn}-m^2)}+\left(\frac{\omega_\lambda}{\omega}\right)^2\right].$$

For TM modes, $\omega_{mn}^{\mathrm{TM}} = x_{mn} c/r$.
For copper, the resistivity is $\rho = \sigma^{-1} = 1.7 \times 10^{-8}\ \Omega \cdot \mathrm{m}$.
Take $\mu_c = \mu_0$. Also $\omega_\lambda = \gamma_\lambda c$. $\delta_\lambda = \sqrt{2/\mu_c \sigma \omega_\lambda}$.
$\epsilon_0 = 8.854 \times 10^{-12}\,\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$, so

$$\begin{aligned}\sqrt{\frac{\epsilon}{\mu}}\frac{1}{\sigma\delta_\lambda} &= \sqrt{\frac{c\epsilon_0\gamma_\lambda}{2\sigma}} = 4.75 \times 10^{-6}\sqrt{\gamma_\lambda}\sqrt{\frac{\mathrm{m}}{\mathrm{s}}\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}\Omega\mathrm{m}} \\ &= 4.75 \times 10^{-6}\ \mathrm{m}^{1/2} \cdot \sqrt{\frac{x_{mn}}{r}}.\end{aligned}$$

The units combine to $\mathrm{m}^{1/2}$ as $\Omega = \dfrac{\mathrm{V}}{\mathrm{A}} = \dfrac{\mathrm{J/C}}{\mathrm{C/s}} = \mathrm{Nms/C}^2$.
In comparison to the TM_{12} mode for a square of side a, we see that $\beta^{\mathrm{TM}} = \frac{a}{2r}\beta_{12}^{\square\mathrm{TM}}$. As the cutoff frequencies are $2.4048c/r$ and $\sqrt{5}\pi c/a$ respectively, we see that the comparable dimensions are $r = (2.4048/\sqrt{5}\pi)a = 0.342a$, much smaller, and then $a/2r = 1.46$, so the smaller pipe does have faster attenuation.

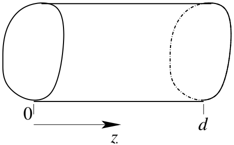
For TE modes, there is an extra factor of

$$\frac{1}{(x'^2_{mn} - m^2)} + \left(\frac{\omega_\lambda}{\omega}\right)^2 .$$

which for the lowest mode is $0.4185 + (\omega_\lambda/\omega)^2$ compared to $0.5 + (\omega_\lambda/\omega)^2$ for the square. But the cutoff frequencies are now $1.841c/r$ and $\sqrt{2}\pi c/a$, so comparable dimensions have $r = 1.841a/\sqrt{2}\pi = 0.414a$.

Resonant Cavities

In infinite cylindrical waveguide, have waves with (angular) frequency ω for each arbitrary definite wavenumber k, with $\omega = c\sqrt{k^2 + \gamma_\lambda^2}$. For each mode λ and each $\omega > \omega_\lambda = c\gamma_\lambda$, there are two modes, $k = \pm\sqrt{\omega^2/c^2 - \gamma_\lambda^2}$.
Standing waves by superposition.
Flat conductors at z=0 and z=d.
For TM, the determining field is



$$E_z = \left(\psi^{(k)} e^{ikz} + \psi^{(-k)} e^{-ikz}\right) e^{-i\omega t},$$

$$\vec{E}_t = i\frac{k}{\gamma_\lambda^2}\vec{\nabla}_t\psi^{(k)} e^{ikz} + i\frac{-k}{\gamma_\lambda^2}\vec{\nabla}_t\psi^{(-k)} e^{-ikz}$$

$\vec{E}_t = 0$ at endcap so $\psi^{(k)} = \psi^{(-k)}$ (at z=0) and $\sin kd = 0$ (at z=d). So $k = p\pi/d$, $p \in \mathbb{Z}$.

Thus the TM fields are

$$\left.\begin{aligned} E_z &= \cos\left(\frac{p\pi z}{d}\right)\ \psi(x,y) \\ \vec{E}_t &= -\frac{p\pi}{d\gamma_\lambda^2}\sin\left(\frac{p\pi z}{d}\right)\vec{\nabla}_t\psi \\ \vec{H}_t &= i\frac{\epsilon\omega}{\gamma_\lambda^2}\cos\left(\frac{p\pi z}{d}\right)\hat{z}\times\vec{\nabla}_t\psi \end{aligned}\right\} \qquad \left\{\begin{array}{c}\text{for TM modes}\\ \text{with } p\in\mathbb{Z}\end{array}\right.$$

Note that in choosing signs we must keep track that half the wave has wavenumber $-k$.

For TE modes, H_z determines all, and must vanish at endcaps (as $\hat{n}\cdot\vec{B}$ vanishes at boundaries). So

$$\left.\begin{aligned} H_z &= \sin\left(\frac{p\pi z}{d}\right)\ \psi(x,y) \\ \vec{H}_t &= \frac{p\pi}{d\gamma_\lambda^2}\cos\left(\frac{p\pi z}{d}\right)\vec{\nabla}_t\psi \\ \vec{E}_t &= -i\frac{\omega\mu}{\gamma_\lambda^2}\sin\left(\frac{p\pi z}{d}\right)\hat{z}\times\vec{\nabla}_t\psi \end{aligned}\right\} \qquad \left\{\begin{array}{c}\text{for TE modes}\\ \text{with } p\in\mathbb{Z}, p\neq 0.\end{array}\right. .$$

Generally the 2D mode λ requires two indices.
For a circular cylinder, we have angular index m, and radial index n specifying which root of J_m (for TM) or of $dJ(x)/dx$ (for TE).

$$\gamma_{mn} = \begin{cases} x_{mn}/R & \text{(TM modes)} \quad J_m(x_{mn}) = 0 \\ x'_{mn}/R & \text{(TE modes)} \quad \frac{dJ_m}{dx}(x'_{mn}) = 0 \end{cases}.$$

with R the radius of the cylinder.

Now we have a third index, p.

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\frac{x_{mn}^2}{R^2} + \frac{p^2\pi^2}{d^2}} \quad \text{with } p \geq 0 \text{ for TM modes,}$$

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\frac{x'^2_{mn}}{R^2} + \frac{p^2\pi^2}{d^2}} \quad \text{with } p > 0 \text{ for TE modes.}$$

Lowest TM mode, $\omega_{010} = cx_{01}/R = 2.405c/R$, independent of d.

For TE modes, $p \neq 0$, so lowest mode with $\gamma = x'_{11}/R$ has

$$\omega_{111} = 1.841\frac{c}{R}\sqrt{1 + 2.912R^2/d^2}.$$

As this depends on d, such a cavity can be tuned by having a movable piston for one endcap.

Power Loss and Quality Factor

What if conductor not perfect? Power losses in sides and in endcaps. Rate is proportional to $U(t)$, the energy stored inside. Let

$$-\Delta U = \text{energy loss per cycle}, \qquad Q := 2\pi U/|\Delta U| \; .$$

One period is $\Delta t = 2\pi/\omega$. Assume $Q \gg 1$, so $|\Delta U| \ll U$, $\Delta U = -2\pi U/Q = (2\pi/\omega)dU/dt$, so

$$U(t) = U(0)e^{-\omega t/Q}.$$

Q is called the resonance "quality factor" or "Q-value". So if an oscillation excited at time $t = 0$ by momentary external influence,

$$U(t) \propto e^{-\omega t/Q} \Longrightarrow E(t) = E_0 e^{-i\omega_0(1-i/2Q)t}\Theta(t),$$

The Heaviside function $\Theta(t) = 1$ for $t > 0$, $= 0$ for $t < 0$. This $\delta(t)$ excitation consists of equal amounts at all frequencies.

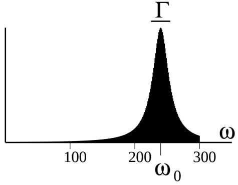
Breit-Wigner

It produces a frequency response

$$\begin{aligned} E(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(t) e^{i\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}} E_0 \int_0^{\infty} e^{i(\omega-\omega_0-i\Gamma/2)t} dt \\ &= \frac{iE_0}{\sqrt{2\pi}} \frac{1}{\omega-\omega_0-i\Gamma/2}, \end{aligned}$$

with $\Gamma := \omega_0/Q$.
$|E(\omega)|^2$ gives the response to excitations of any frequency, with

$$|E(\omega)|^2 \propto \frac{1}{(\omega-\omega_0)^2+\Gamma^2/4}.$$



This is called the Breit-Wigner response. Γ is mistakenly called the half-width. Really full-width at half-maximum.

Calculation of power loss as for waveguide, but need to include power loss in endcaps as well. Jackson, pp 373-374. We will skip this.

Earth and Ionosphere:

Not all cavities cylindrical. Consider surface of Earth, and ionosphere, an ionized layer about 100 km up. Concentric conducting spheres acting as endcaps, of a waveguide with no walls, but topology!

Need spherical coordinates, of course. More generally, may need other curvilinear coordinates (as you will for your projects).
So we will digress to discuss curvilinear coordinates.