Power Loss

How much power is dissipated (per unit area?). 2 ways:
1) Flow of energy into conductor: Energy flow given by $\vec{S} = \vec{E} \times \vec{H}$, for real fields $\vec{E}$ and $\vec{H}$.
so[1] $\langle \vec{S} \rangle = \frac{1}{2}\mathrm{Re}\left(\vec{E} \times \vec{H}^*\right)$, and $dP_{\text{loss}}/dA = -\hat{n} \cdot \langle \vec{S} \rangle$, so

$$\begin{aligned} \frac{dP_{\text{loss}}}{dA} &= -\frac{1}{2}\sqrt{\frac{\mu_c \omega}{2\sigma}}\hat{n} \cdot \mathrm{Re}\left[(1-i)(\hat{n} \times \vec{H}_\parallel) \times \vec{H}_\parallel^*\right] \\ &= \frac{\mu_c \omega \delta}{4}|\vec{H}_\parallel|^2 = \frac{1}{2\sigma\delta}|\vec{H}_\parallel|^2 \end{aligned}$$

Method 2, Ohmic heating, power lost per unit volume $\frac{1}{2}\vec{J} \cdot \vec{E}^* = |\vec{J}|^2/2\sigma$, $|\vec{J}| = \sigma \vec{E}_c = \frac{\sqrt{2}}{\delta}|\vec{H}_\parallel| e^{-\xi/\delta}$, the power loss per unit area is

$$\frac{dP_{\text{loss}}}{dA} = \frac{1}{\delta^2\sigma}|\vec{H}_\parallel|^2 \int_0^\infty d\xi\, e^{-2\xi/\delta} = \frac{1}{2\delta\sigma}|\vec{H}_\parallel|^2.$$

Agrees with method 1.

[1] The $\frac{1}{2}$, Re, and $*$ discussed in lectures B and H.

In terms of surface current

$$\begin{aligned} \vec{K}_{\text{eff}} &= \int_0^\infty d\xi\, \vec{J}(\xi) = \frac{1}{\delta}\hat{n} \times \vec{H}_\parallel \int_0^\infty d\xi\, (1-i)e^{-\xi(1-i)/\delta} \\ &= \hat{n} \times \vec{H}_\parallel. \end{aligned}$$

Thus

$$\frac{dP_{\text{loss}}}{dA} = \frac{1}{2\sigma\delta}|\vec{K}_{\text{eff}}|^2.$$

$\dfrac{1}{\sigma\delta}$ is surface resistance (per unit area) and $\dfrac{\vec{E}_\parallel}{\vec{K}_{\text{eff}}} = \dfrac{1-i}{\sigma\delta}$ is the surface impediance Z.

Wave Guides

For electromagnetic fields with a fixed geometry of linear materials, fourier transform decouples, and we can work with frequency modes,

$$\begin{aligned} \vec{E}(\vec{x}, t) &= \vec{E}(x, y, z)\, e^{-i\omega t} \\ \vec{B}(\vec{x}, t) &= \vec{B}(x, y, z)\, e^{-i\omega t} \end{aligned}$$

Actually the fields are the real parts of these complex expressions.
If $\rho = 0$, $\vec{J} = 0$, Maxwell gives

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = i\omega\vec{B}, \qquad \vec{\nabla} \cdot \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{B} = 0,$$

$$\vec{\nabla} \times \vec{B} = \mu\vec{\nabla} \times \vec{H} = \mu\frac{\partial \vec{D}}{\partial t} = \mu\epsilon\frac{\partial \vec{E}}{\partial t} = -i\omega\mu\epsilon\vec{E}.$$

Then

$$\nabla^2\vec{E} = -\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) = -\vec{\nabla}\times(i\omega\vec{B}) = -\omega^2\mu\epsilon\vec{E}.$$

and similarly for $\vec{B}$, so we get Helmholtz equations

$$\left(\nabla^2 + \omega^2\mu\epsilon\right)\vec{E} = 0, \qquad \left(\nabla^2 + \omega^2\mu\epsilon\right)\vec{B} = 0.$$

Consider a waveguide, a cylinder of arbitrary cross section but uniform in z. Fourier transform in z

$$\begin{aligned} \vec{E}(x, y, z, t) &= \vec{E}(x, y)e^{ikz - i\omega t} \\ \vec{B}(x, y, z, t) &= \vec{B}(x, y)e^{ikz - i\omega t} \end{aligned}$$

k can take either sign (and a standing wave is a superposition of $k = \pm|k|$). The Helmholtz equations give

$$\left[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)\right]\begin{pmatrix}\vec{E}(x, y) \\ \vec{B}(x, y)\end{pmatrix} = 0, \quad \nabla_t^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Decompose longitudinal and transverse

Let

$$\begin{matrix} \vec{E} = E_z\hat{z} + \vec{E}_t \\ \vec{B} = B_z\hat{z} + \vec{B}_t \end{matrix} \quad \text{with} \quad \begin{matrix} \vec{E}_t \perp \hat{z} \\ \vec{B}_t \perp \hat{z} \end{matrix}$$

$$\begin{aligned} (\vec{\nabla} \times \vec{E})_z &= (\vec{\nabla}_t \times \vec{E}_t)_z = i\omega B_z, \\ (\vec{\nabla} \times \vec{E})_\perp &= \hat{z} \times \frac{\partial \vec{E}_t}{\partial z} - \hat{z} \times \nabla_t E_z = i\omega\vec{B}_t. \end{aligned}$$

For any vector $\vec{V}$, $\hat{z} \times (\hat{z} \times \vec{V}) = -\vec{V} + \hat{z}(\hat{z} \cdot V)$, so for a transverse vector $\hat{z} \times (\hat{z} \times \vec{V}_t) = -\vec{V}_t$. Taking $\hat{z} \times$ last equation,

$$\frac{\partial \vec{E}_t}{\partial z} - \vec{\nabla}_t E_z = -i\omega\hat{z} \times \vec{B}_t. \tag{1}$$

Similarly decomposition of $\vec{\nabla} \times \vec{B} = -i\omega\mu\epsilon\vec{E}$ gives

$$\begin{aligned} \left(\vec{\nabla}_t \times \vec{B}_t\right)_z &= -i\omega\mu\epsilon E_z \\ \frac{\partial \vec{B}_t}{\partial z} - \vec{\nabla}_t B_z &= i\omega\mu\epsilon\hat{z} \times \vec{E}_t. \end{aligned} \tag{2}$$

Divergencelessness:

$$\vec{\nabla}_t \cdot \vec{E}_t + \frac{\partial E_z}{\partial z} = 0, \qquad \vec{\nabla}_t \cdot \vec{B}_t + \frac{\partial B_z}{\partial z} = 0.$$

Equations (1) and (2), with the fourier transform in z, give

$$\begin{aligned} ik\vec{E}_t + i\omega\hat{z} \times \vec{B}_t &= \vec{\nabla}_t E_z & (3) \\ ik\vec{B}_t - i\omega\mu\epsilon\hat{z} \times \vec{E}_t &= \vec{\nabla}_t B_z & (4) \end{aligned}$$

Solving 4 for $\vec{B}_t$ and plugging into 3, and then the reverse for $\vec{E}_t$, give

$$E_t = i\frac{k\vec{\nabla}_t E_z - \omega\hat{z} \times \vec{\nabla}_t B_z}{\omega^2\mu\epsilon - k^2} \tag{5}$$

$$B_t = i\frac{k\vec{\nabla}_t B_z + \omega\mu\epsilon\hat{z} \times \vec{\nabla}_t E_z}{\omega^2\mu\epsilon - k^2} \tag{6}$$

Unless $k^2 = k_0^2 := \mu\epsilon\omega^2$, E_z and B_z determine the rest.

We have seen that E_z and B_z largely determine the fields, and these satisfy the two-dimensional Helmholtz equation

$$\left(\nabla_t^2 + \gamma^2\right)\psi = 0 \qquad \text{with} \quad \gamma^2 = \mu\epsilon\omega^2 - k^2 \tag{7}$$

If the walls of the waveguide are very good conductors, we may impose the perfect conductor conditions $E_\parallel \approx 0$ and $B_\perp \approx 0$ on the boundary S of the two-dimensional cross section. E_z is parallel to the boundary so $E_z|_S = 0$. Also the component of $\vec{E}_t$ parallel to the boundary vanishes at the wall, so $\vec{E}_t$ is in the $\pm\hat{n}$ direction. Then from the $\hat{n}$ component of (2) (normal to the boundary)

$$\frac{\partial \hat{n}\cdot\vec{B}_t}{\partial z} - \hat{n}\cdot\vec{\nabla}_t B_z = i\omega\mu\epsilon\hat{n}\cdot\left(\hat{z}\times\vec{E}_t\right) \Longrightarrow 0 - \frac{\partial B_z}{\partial n} = 0,$$

where $\partial/\partial n$ is the derivative normal to the surface. So we have Dirichlet conditions on E_z and Neumann conditions for B_z.

In general, nonzero solutions exist only for discrete values of γ, and those values are generally different for Dirichlet and for Neumann. So we need to consider

- TEM modes, with $E_z(x,y) = B_z(x,y) \equiv 0$. That is, there are no longitudinal fields, both electric (E) and magnetic (M) fields are purely transverse to the direction z of propagation.
- TE modes, $E_z(x,y) \equiv 0$, and the transverse fields are determined by the gradiant of $B_z = \psi$, a solution of (7) with Neumann conditions.
- TM modes, $B_z(x,y) \equiv 0$, and the transverse fields are determined by $E_z = \psi$, a solution of (7) with zero boundary conditions.

TEM modes

With $E_z(x,y) = B_z(x,y) \equiv 0$, (5) and (6) $\Longrightarrow$ everything vanishes or the denominator vanishes,

$$k = \pm k_0 \qquad \text{with} \quad k_0 = \sqrt{\mu\epsilon}\,\omega$$

Wave travels $\parallel z$ with speed $1/\sqrt{\mu\epsilon}$, same as for infinite medium. No dispersion.
$\vec{\nabla}_t\cdot\vec{E}_t = 0$ and $\vec{\nabla}_t\times\vec{E}_t = i\omega B_z = 0$, so $\exists\Phi \ni \vec{E}_t = -\vec{\nabla}_t\Phi$ (though Φ might not be single valued) and $\nabla^2\Phi = 0$.
As $\vec{E}_\parallel\Big|_S = 0$, $\Phi =$ constant on each boundary. If cross section simply connected, $\Phi =$ constant, $\vec{E} = 0$
No TEM modes on simply connected cylinder
Yes TEM modes on coaxial cable, or two parallel wires.

TE and TM modes

Equations (5) and (6) simplify for

- TM modes, $B_z = 0$, $\quad \gamma^2\vec{E}_t = ik\vec{\nabla}_t E_z$,
  $\gamma^2\vec{B}_t = i\mu\epsilon\omega\hat{z}\times\vec{\nabla}_t E_z$, $\quad$ so $\vec{H}_t = \epsilon\omega k^{-1}\hat{z}\times\vec{E}_t$.
- TE modes, $E_z = 0$, $\quad \gamma^2\vec{B}_t = ik\vec{\nabla}_t B_z$,
  $\gamma^2\vec{E}_t = -i\omega\hat{z}\times\vec{\nabla}_t B_z$, $\quad$ so

$$\vec{E}_t = -\omega\hat{z}\times\vec{B}_t/k \underset{\hat{z}\times}{\Longrightarrow} H_t = k\hat{z}\times E_t/\mu\omega.$$

In either case, $\vec{H}_t = \frac{1}{Z}\hat{z}\times\vec{E}_t$, with

$$Z = \begin{cases} k/\epsilon\omega = (k/k_0)\sqrt{\mu/\epsilon} & \text{TM} \\ \mu\omega/k = (k_0/k)\sqrt{\mu/\epsilon} & \text{TE} \end{cases}$$

To Summarize

Solutions given by $\psi(x,y)$, with $\left(\nabla_t^2 + \gamma^2\right)\psi = 0$, $\gamma^2 = \mu\epsilon\omega^2 - k^2$, by

$$\begin{array}{ll} \text{TM:} & E_z = \psi e^{ikz - i\omega t}, \quad \vec{E}_t = ik\gamma^{-2}\vec{\nabla}_t\psi e^{ikz-i\omega t} \\ & \text{with} \quad \psi|_\Gamma = 0 \\ \text{TE:} & H_z = \psi e^{ikz - i\omega t}, \quad \vec{H}_t = ik\gamma^{-2}\vec{\nabla}_t\psi e^{ikz-i\omega t} \\ & \text{with} \quad \hat{n}\cdot\vec{\nabla}_t\psi|_\Gamma = 0 \end{array}$$

By looking at $0 = \int_A \psi^*\left(\nabla_t^2 + \gamma^2\right)\psi$ we can show $\gamma^2 \geq 0$. There are solutions for **discrete** values γ_λ, so only certain wave numbers k_λ for a given frequency can propagate:

$$k_\lambda^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2,$$

and only frequencies $\omega > \omega_\lambda := \gamma_\lambda/\sqrt{\mu\epsilon}$ can propagate, and $k_\lambda < \sqrt{\mu\epsilon}\,\omega$, the infinite medium wavenumber. Phase velocity $v_p = \omega/k_\lambda$ is greater than in the infinite medium.

Example: Circular Wave Guide

Jackson does rectangle. You should too. Needed to do homework.

We will consider a circular pipe of (inner) radius r. Of course we should use polar coordinates ρ, ϕ, with

$$\nabla_t^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}, \qquad \text{try} \quad \psi(\rho,\phi) = R(\rho)\Phi(\phi),$$

$$\left(\nabla_t^2 + \gamma^2\right)\psi = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2 R(\rho)\right)\Phi(\phi) + \frac{1}{\rho^2}R(\rho)\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = 0.$$

Divide by $R(\rho)\Phi(\phi)$ and multiply by ρ^2:

$$\frac{1}{R(\rho)}\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2\rho^2 R(\rho)\right) + \frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = 0.$$

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Divide by $R(\rho)\Phi(\phi)$ and multiply by ρ^2:

$$\frac{1}{R(\rho)}\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2\rho^2 R(\rho)\right) = C$$

$$\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = -C.$$

Solving it

Φ first:

$$\frac{\partial^2\Phi(\phi)}{\partial\phi^2} + C\Phi(\phi) = 0$$

$\Phi(\phi) = e^{\pm i\sqrt{C}\phi}$. Periodicity $\Longrightarrow \sqrt{C} = m \in \mathbb{Z}$.

Now $R(\rho)$:

$$\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \gamma^2\rho^2 - m^2\right)R(\rho) = 0$$

Bessel equation, solutions regular at origin are

$$R(\rho) \propto J_m(\gamma\rho), \quad \text{so} \quad \psi(\rho,\phi) = \sum_{m,n} A_{m,n} J_m(\gamma_{mn}\rho) e^{im\phi}.$$

γ_{mn} is determined by boundary conditions...

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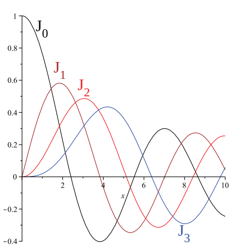
Shapiro
Wave Guides
Power Loss
Wave Guides
TEM, TE, and TM Modes
TEM modes
Example

Boundary conditions:

For TM, $\psi(r,\phi) = 0 \Longrightarrow J_m(\gamma r) = 0$, so $\gamma_{mn}^{\rm TM} = x_{mn}/r$ where x_{mn} is the n'th value of $x > 0$ for which $J_m(x) = 0$, given on page 114.

For TE, $\hat{n} \cdot \vec{\nabla}_t \psi(r,\phi) = 0 \Longrightarrow \frac{dJ_m}{dr}(\gamma r) = 0$, so $\gamma_{mn}^{\rm TM} = x'_{mn}/r$ where x'_{mn} is the n'th value of $x > 0$ for which $dJ_m(x)/dx = 0$, given on page 370.



Thus the lowest cutoff frequency is the $m = 1$ TE mode, with $x'_{11} = 1.841$ while the lowest TM mode or circularly symmetric mode has $x_{01} = 2.405$.

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Shapiro
Wave Guides
Power Loss
Wave Guides
TEM, TE, and TM Modes
TEM modes
Example

For a waveguide 5 cm in diameter, with air or vacuum inside, the cutoff frequencies are $f = \frac{\omega}{2\pi} = 3.5$ GHz for the lowest TE and 4.6 GHz for the lowest TM modes.

Physics 504, Spring 2011 Electricity and Magnetism
Shapiro
Wave Guides
Power Loss
Wave Guides
TEM, TE, and TM Modes
TEM modes
Example