Power Loss

How much power is dissipated (per unit area?). 2 ways:
1) Flow of energy into conductor: Energy flow given by $\vec{S} = \vec{E} \times \vec{H}$, for real fields $\vec{E}$ and $\vec{H}$.
so[1] $\langle \vec{S} \rangle = \frac{1}{2}\mathrm{Re}\left(\vec{E} \times \vec{H}^*\right)$, and $dP_{\text{loss}}/dA = -\hat{n} \cdot \langle \vec{S} \rangle$, so

$$\begin{aligned} \frac{dP_{\text{loss}}}{dA} &= -\frac{1}{2}\sqrt{\frac{\mu_c \omega}{2\sigma}}\hat{n} \cdot \mathrm{Re}\left[(1-i)(\hat{n} \times \vec{H}_{\parallel}) \times \vec{H}_{\parallel}^*\right] \\ &= \frac{\mu_c \omega \delta}{4}|\vec{H}_{\parallel}|^2 = \frac{1}{2\sigma\delta}|\vec{H}_{\parallel}|^2 \end{aligned}$$

Method 2, Ohmic heating, power lost per unit volume $\frac{1}{2}\vec{J} \cdot \vec{E}^* = |\vec{J}|^2/2\sigma$, $|\vec{J}| = \sigma \vec{E}_c = \frac{\sqrt{2}}{\delta}|\vec{H}_{\parallel}|e^{-\xi/\delta}$, the power loss per unit area is

$$\frac{dP_{\text{loss}}}{dA} = \frac{1}{\delta^2 \sigma}|\vec{H}_{\parallel}|^2 \int_0^\infty d\xi \, e^{-2\xi/\delta} = \frac{1}{2\delta\sigma}|\vec{H}_{\parallel}|^2.$$

Agrees with method 1.

[1] The $\frac{1}{2}$, Re, and $*$ discussed in lectures B and H.

In terms of surface current

$$\begin{aligned}\vec{K}_{\text{eff}} &= \int_0^\infty d\xi\, \vec{J}(\xi) = \frac{1}{\delta}\hat{n} \times \vec{H}_{\parallel} \int_0^\infty d\xi\, (1-i)e^{-\xi(1-i)/\delta} \\ &= \hat{n} \times \vec{H}_{\parallel}.\end{aligned}$$

Thus

$$\frac{dP_{\text{loss}}}{dA} = \frac{1}{2\sigma\delta}|\vec{K}_{\text{eff}}|^2.$$

$\frac{1}{\sigma\delta}$ is surface resistance (per unit area) and $\frac{\vec{E}_{\parallel}}{\vec{K}_{\text{eff}}} = \frac{1-i}{\sigma\delta}$ is the surface impediance Z.

Wave Guides

For electromagnetic fields with a fixed geometry of linear materials, fourier transform decouples, and we can work with frequency modes,

$$\begin{aligned}\vec{E}(\vec{x},t) &= \vec{E}(x,y,z)\, e^{-i\omega t}\\ \vec{B}(\vec{x},t) &= \vec{B}(x,y,z)\, e^{-i\omega t}\end{aligned}$$

Actually the fields are the real parts of these complex expressions.
If $\rho = 0$, $\vec{J} = 0$, Maxwell gives

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = i\omega\vec{B}, \qquad \vec{\nabla}\cdot\vec{E} = 0, \qquad \vec{\nabla}\cdot\vec{B} = 0,$$

$$\vec{\nabla}\times\vec{B} = \mu\vec{\nabla}\times\vec{H} = \mu\frac{\partial\vec{D}}{\partial t} = \mu\epsilon\frac{\partial\vec{E}}{\partial t} = -i\omega\mu\epsilon\vec{E}.$$

Then

$$\nabla^2\vec{E} = -\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) = -\vec{\nabla}\times(i\omega\vec{B}) = -\omega^2\mu\epsilon\vec{E}.$$

and similarly for $\vec{B}$, so we get Helmholtz equations

$$\left(\nabla^2 + \omega^2\mu\epsilon\right)\vec{E} = 0, \qquad \left(\nabla^2 + \omega^2\mu\epsilon\right)\vec{B} = 0.$$

Consider a waveguide, a cylinder of arbitrary cross section but uniform in z. Fourier transform in z

$$\begin{aligned}\vec{E}(x,y,z,t) &= \vec{E}(x,y)e^{ikz-i\omega t}\\ \vec{B}(x,y,z,t) &= \vec{B}(x,y)e^{ikz-i\omega t}\end{aligned}$$

k can take either sign (and a standing wave is a superposition of $k = \pm|k|$). The Helmholtz equations give

$$\left[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)\right]\begin{pmatrix}\vec{E}(x,y)\\ \vec{B}(x,y)\end{pmatrix} = 0, \quad \nabla_t^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Decompose longitudinal and transverse

Let

$$\vec{E} = E_z\hat{z} + \vec{E}_t \quad \text{with} \quad \vec{E}_t \perp \hat{z}$$
$$\vec{B} = B_z\hat{z} + \vec{B}_t \qquad\qquad \vec{B}_t \perp \hat{z}$$

$$\begin{aligned}(\vec{\nabla}\times\vec{E})_z &= (\vec{\nabla}_t\times\vec{E}_t)_z = i\omega B_z,\\ (\vec{\nabla}\times\vec{E})_\perp &= \hat{z}\times\frac{\partial\vec{E}_t}{\partial z} - \hat{z}\times\nabla_t E_z = i\omega\vec{B}_t.\end{aligned}$$

For any vector $\vec{V}$, $\hat{z}\times(\hat{z}\times\vec{V}) = -\vec{V} + \hat{z}(\hat{z}\cdot V)$, so for a transverse vector $\hat{z}\times(\hat{z}\times\vec{V}_t) = -\vec{V}_t$. Taking $\hat{z}\times$ last equation,

$$\frac{\partial\vec{E}_t}{\partial z} - \vec{\nabla}_t E_z = -i\omega\hat{z}\times\vec{B}_t. \tag{1}$$

Similarly decomposition of $\vec{\nabla}\times\vec{B} = -i\omega\mu\epsilon\vec{E}$ gives

$$\begin{aligned}\left(\vec{\nabla}_t\times\vec{B}_t\right)_z &= -i\omega\mu\epsilon E_z\\ \frac{\partial\vec{B}_t}{\partial z} - \vec{\nabla}_t B_z &= i\omega\mu\epsilon\hat{z}\times\vec{E}_t. \qquad (2)\end{aligned}$$

Divergencelessness:

$$\vec{\nabla}_t \cdot \vec{E}_t + \frac{\partial E_z}{\partial z} = 0, \qquad \vec{\nabla}_t \cdot \vec{B}_t + \frac{\partial B_z}{\partial z} = 0.$$

Equations (1) and (2), with the fourier transform in z, give

$$ik\vec{E}_t + i\omega\hat{z} \times \vec{B}_t = \vec{\nabla}_t E_z \tag{3}$$

$$ik\vec{B}_t - i\omega\mu\epsilon\hat{z} \times \vec{E}_t = \vec{\nabla}_t B_z \tag{4}$$

Solving 4 for $\vec{B}_t$ and plugging into 3, and then the reverse for $\vec{E}_t$, give

$$E_t = i\frac{k\vec{\nabla}_t E_z - \omega\hat{z} \times \vec{\nabla}_t B_z}{\omega^2\mu\epsilon - k^2} \tag{5}$$

$$B_t = i\frac{k\vec{\nabla}_t B_z + \omega\mu\epsilon\hat{z} \times \vec{\nabla}_t E_z}{\omega^2\mu\epsilon - k^2} \tag{6}$$

Unless $k^2 = k_0^2 := \mu\epsilon\omega^2$, E_z and B_z determine the rest.

We have seen that E_z and B_z largely determine the fields, and these satisfy the two-dimensional Helmholtz equation

$$\left(\nabla_t^2 + \gamma^2\right)\psi = 0 \qquad \text{with} \quad \gamma^2 = \mu\epsilon\omega^2 - k^2 \tag{7}$$

If the walls of the waveguide are very good conductors, we may impose the perfect conductor conditions $E_\parallel \approx 0$ and $B_\perp \approx 0$ on the boundary S of the two-dimensional cross section. E_z is parallel to the boundary so $E_z|_S = 0$. Also the component of $\vec{E}_t$ parallel to the boundary vanishes at the wall, so $\vec{E}_t$ is in the $\pm\hat{n}$ direction. Then from the $\hat{n}$ component of (2) (normal to the boundary)

$$\frac{\partial \hat{n}\cdot\vec{B}_t}{\partial z} - \hat{n}\cdot\vec{\nabla}_t B_z = i\omega\mu\epsilon\hat{n}\cdot\left(\hat{z}\times\vec{E}_t\right) \Longrightarrow 0 - \frac{\partial B_z}{\partial n} = 0,$$

where $\partial/\partial n$ is the derivative normal to the surface. So we have Dirichlet conditions on E_z and Neumann conditions for B_z.

In general, nonzero solutions exist only for discrete values of γ, and those values are generally different for Dirichlet and for Neumann. So we need to consider

- TEM modes, with $E_z(x,y) = B_z(x,y) \equiv 0$. That is, there are no longitudinal fields, both electric (E) and magnetic (M) fields are purely transverse to the direction z of propagation.
- TE modes, $E_z(x,y) \equiv 0$, and the transverse fields are determined by the gradiant of $B_z = \psi$, a solution of (7) with Neumann conditions.
- TM modes, $B_z(x,y) \equiv 0$, and the transverse fields are determined by $E_z = \psi$, a solution of (7) with zero boundary conditions.

TEM modes

With $E_z(x,y) = B_z(x,y) \equiv 0$, (5) and (6) $\Longrightarrow$ everything vanishes or the denominator vanishes,

$$k = \pm k_0 \qquad \text{with} \quad k_0 = \sqrt{\mu\epsilon}\,\omega$$

Wave travels $\parallel z$ with speed $1/\sqrt{\mu\epsilon}$, same as for infinite medium. No dispersion.
$\vec{\nabla}_t \cdot \vec{E}_t = 0$ and $\vec{\nabla}_t \times \vec{E}_t = i\omega B_z = 0$, so $\exists \Phi \ni \vec{E}_t = -\vec{\nabla}_t \Phi$ (though Φ might not be single valued) and $\nabla^2\Phi = 0$.
As $\vec{E}_\parallel\Big|_S = 0$, $\Phi =$ constant on each boundary. If cross section simply connected, $\Phi =$ constant, $\vec{E} = 0$
No TEM modes on simply connected cylinder
Yes TEM modes on coaxial cable, or two parallel wires.

TE and TM modes

Equations (5) and (6) simplify for

- TM modes, $B_z = 0$, $\gamma^2 \vec{E}_t = ik\vec{\nabla}_t E_z$, $\gamma^2 \vec{B}_t = i\mu\epsilon\omega\hat{z} \times \vec{\nabla}_t E_z$, so $\vec{H}_t = \epsilon\omega k^{-1}\hat{z} \times \vec{E}_t$.
- TE modes, $E_z = 0$, $\gamma^2 \vec{B}_t = ik\vec{\nabla}_t B_z$, $\gamma^2 \vec{E}_t = -i\omega\hat{z} \times \vec{\nabla}_t B_z$, so

$$\vec{E}_t = -\omega\hat{z} \times \vec{B}_t/k \underset{\hat{z}\times}{\Longrightarrow} H_t = k\hat{z} \times E_t/\mu\omega.$$

In either case, $\vec{H}_t = \frac{1}{Z}\hat{z} \times \vec{E}_t$, with

$$Z = \begin{cases} k/\epsilon\omega = (k/k_0)\sqrt{\mu/\epsilon} & \text{TM} \\ \mu\omega/k = (k_0/k)\sqrt{\mu/\epsilon} & \text{TE} \end{cases}$$

To Summarize

Solutions given by $\psi(x,y)$, with $\left(\nabla_t^2+\gamma^2\right)\psi=0$, $\gamma^2=\mu\epsilon\omega^2-k^2$, by

$$\begin{array}{ll}
\text{TM:} & E_z=\psi e^{ikz-i\omega t}, \quad \vec{E}_t=ik\gamma^{-2}\vec{\nabla}_t\psi e^{ikz-i\omega t} \\
& \text{with} \quad \psi|_\Gamma=0 \\
\text{TE:} & H_z=\psi e^{ikz-i\omega t}, \quad \vec{H}_t=ik\gamma^{-2}\vec{\nabla}_t\psi e^{ikz-i\omega t} \\
& \text{with} \quad \hat{n}\cdot\vec{\nabla}_t\psi|_\Gamma=0
\end{array}$$

By looking at $0=\int_A\psi^*\left(\nabla_t^2+\gamma^2\right)\psi$ we can show $\gamma^2\geq 0$. There are solutions for **discrete** values γ_λ, so only certain wave numbers k_λ for a given frequency can propagate:

$$k_\lambda^2=\mu\epsilon\omega^2-\gamma_\lambda^2,$$

and only frequencies $\omega>\omega_\lambda:=\gamma_\lambda/\sqrt{\mu\epsilon}$ can propagate, and $k_\lambda<\sqrt{\mu\epsilon}\,\omega$, the infinite medium wavenumber. Phase velocity $v_p=\omega/k_\lambda$ is greater than in the infinite medium.

Example: Circular Wave Guide

Jackson does rectangle. You should too. Needed to do homework.

We will consider a circular pipe of (inner) radius r. Of course we should use polar coordinates ρ, ϕ, with

$$\nabla_t^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}, \qquad \text{try} \quad \psi(\rho,\phi) = R(\rho)\Phi(\phi),$$

$$\left(\nabla_t^2 + \gamma^2\right)\psi = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2 R(\rho)\right)\Phi(\phi) + \frac{1}{\rho^2}R(\rho)\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = 0.$$

Divide by $R(\rho)\Phi(\phi)$ and multiply by ρ^2:

$$\frac{1}{R(\rho)}\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2\rho^2 R(\rho)\right) + \frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = 0.$$

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Divide by $R(\rho)\Phi(\phi)$ and multiply by ρ^2:

$$\begin{aligned}\frac{1}{R(\rho)}\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2\rho^2 R(\rho)\right) &= C \\ \frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} &= -C.\end{aligned}$$

Solving it

Φ first:

$$\frac{\partial^2\Phi(\phi)}{\partial\phi^2} + C\Phi(\phi) = 0$$

$\Phi(\phi) = e^{\pm i\sqrt{C}\phi}$. Periodicity $\Longrightarrow \sqrt{C} = m \in \mathbb{Z}$.

Now $R(\rho)$:

$$\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \gamma^2\rho^2 - m^2\right) R(\rho) = 0$$

Bessel equation, solutions regular at origin are

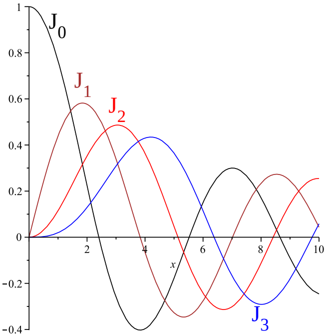
$$R(\rho) \propto J_m(\gamma\rho), \quad \text{so} \quad \psi(\rho,\phi) = \sum_{m,n} A_{m,n} J_m(\gamma_{mn}\rho) e^{im\phi}.$$

γ_{mn} is determined by boundary conditions...

Boundary conditions:

For TM, $\psi(r,\phi) = 0 \implies J_m(\gamma r) = 0$, so $\gamma^{\rm TM}_{mn} = x_{mn}/r$ where x_{mn} is the n'th value of $x > 0$ for which $J_m(x) = 0$, given on page 114.

For TE, $\hat{n} \cdot \vec{\nabla}_t \psi(r,\phi) = 0 \implies \frac{dJ_m}{dr}(\gamma r) = 0$, so $\gamma^{\rm TM}_{mn} = x'_{mn}/r$ where x'_{mn} is the n'th value of $x > 0$ for which $dJ_m(x)/dx = 0$, given on page 370.



Thus the lowest cutoff frequency is the $m = 1$ TE mode, with $x'_{11} = 1.841$ while the lowest TM mode or circularly symmetric mode has $x_{01} = 2.405$.

For a waveguide 5 cm in diameter, with air or vacuum inside, the cutoff frequencies are $f = \frac{\omega}{2\pi} = 3.5$ GHz for the lowest TE and 4.6 GHz for the lowest TM modes.