Causality

We have seen that the issue of how ϵ, μ and n depend on ω raises questions about causality: Can signals travel faster than c, or even backwards in time?
It is very often useful to assume that polarization is linear and local in space, and the polarizability is not time dependent, meaning

$$\vec{D}(\vec{x},\omega) = \epsilon(\omega)\vec{E}(\vec{x},\omega),$$

but that does not mean it is local in time, for

$$\begin{aligned}\vec{D}(\vec{x},t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\,\vec{D}(\vec{x},\omega)e^{-i\omega t}\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\,e^{-i\omega t}\epsilon(\omega)\vec{E}(\vec{x},\omega)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,e^{-i\omega t}\epsilon(\omega)\int_{-\infty}^{\infty} dt'\vec{E}(\vec{x},t')e^{i\omega t'}\end{aligned}$$

Let us write $\epsilon(\omega) = \epsilon_0\left[1+\chi_e(\omega)\right]$ in terms of the electric susceptibility.

$G(t-t')$

Let $G(\tau)$ be the fourier transform of $\chi_e(\omega)$,

$$G(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,e^{-i\omega\tau}\chi_e(\omega).$$

Then we have the relation between $\vec{D}$ and $\vec{E}$ given by

$$\vec{D}(\vec{x},t) = \epsilon_0\left\{\vec{E}(\vec{x},t) + \int_{-\infty}^{\infty} d\tau\,G(\tau)\vec{E}(\vec{x},t-\tau)\right\}.$$

Thus we see that $\vec{D}(\vec{x},t)$ depends linearly on the **function** $\vec{E}(\vec{x},t')$ of time t', but not on the single value $\vec{E}(\vec{x},t)$. That is, the dependence is **non-local** in time. Of course if $\epsilon(\omega)$ were constant, $G(\tau)\propto\delta(\tau)$ and we would have the local $\vec{D}(\vec{x},t) = \epsilon\vec{E}(\vec{x},t)$, but that is not the case generally.

As $\vec{D}(\vec{x},t)$ and $\vec{E}(\vec{x},t)$ are both real, $\vec{D}^*(\vec{x},-\omega) = \vec{D}(\vec{x},\omega)$ and similarly for $\vec{E}$, so for real ω, $\epsilon^*(\omega) = \epsilon(-\omega)$. This means $G(\tau)$ is real (for real τ).

Causality

That the polarization at time t might depend on the electric field at some earlier time t' is not surprising, but shouldn't it be blind to fields at later times? That is, shouldn't we insist $G(\tau) = 0$ for $\tau < 0$)?

Let's consider our oscillator strength model, with

$$\chi_e(\omega) = \frac{\omega_P^2}{\omega_0^2-\omega^2-i\gamma\omega}$$

(or a sum of such contributions with different ω_0's and γ's). Then

$$\begin{aligned}G(\tau) &= \frac{\omega_P^2}{2\pi}\int_{-\infty}^{\infty} d\omega\,\frac{e^{-i\omega\tau}}{\omega_0^2-\omega^2-i\gamma\omega}\\ &= \frac{\omega_P^2}{4\pi\nu_0}\int_{-\infty}^{\infty} d\omega\left(\frac{e^{-i\omega\tau}}{\omega+\nu_0+i\gamma/2} - \frac{e^{-i\omega\tau}}{\omega-\nu_0+i\gamma/2}\right).\end{aligned}$$

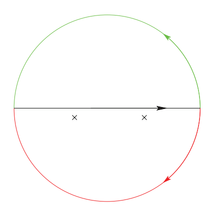
where $\nu_0 = \sqrt{\omega_0^2-\gamma^2/4}$.

Evaluating $G(\tau)$ for oscillators

The integrals can be easily done by closing the contours in the complex plane.

The integral

$$\int_{-\infty}^{\infty} d\omega\,\frac{e^{-i\omega\tau}}{\omega\pm\nu_0+i\gamma/2}$$



can be done by closing the contour with a large semicircle in the upper or lower half plane, whichever gives zero contribution because of the exponential. For negative τ, $|e^{-i\omega\tau}| = e^{-|\tau|\mathrm{Im}\omega}$, so the contribution of the green infinite-radius semicircle in the upper half plane vanishes, and as the integrand is analytic in the enclosed region, the integral is zero. Thus we do have $G(\tau) = 0$ for $\tau < 0$.

For $\tau > 0$, the contour is closed in the lower half plane, and the integral is given by $-2\pi i$ times the sum of the residues, which are $e^{\pm i\nu_0\tau-\gamma\tau/2}$. So the two terms for positive τ fill in the result:

$$G(\tau) = \omega_P^2\frac{\sin(\nu_0\tau)}{\nu_0}\Theta(\tau).$$

Typical values for the lifetime of states, and hence the line-widths of the photons emitted, give γ_i from 10^7/s to 10^9/s, so the effective τ's are of the order of nanoseconds. While the response of $\vec{D}$ to $\vec{E}$ is not instantaneous, it is quick.

Model-independent ϵ's

We will now find constraints on possible forms of $\epsilon(\omega)$ without assumptions on the model of molecular behavior. We have seen that reality of the fields in the time domain requires $\epsilon^*(\omega) = \epsilon(-\omega)$ for real ω. We have only defined and used $\epsilon(\omega)$ for real ω, but if we would like to continue ϵ as a complex valued analytic function, we need to extend the constraint to

$$\epsilon^*(\omega^*) = \epsilon(-\omega), \qquad \chi_e^*(\omega^*) = \chi_e(-\omega).$$

We will also insist that $G(\tau)$ is finite and real for positive τ and zero for negative τ. We might expect $G \underset{\tau\to\infty}{\longrightarrow} 0$ which is true for dielectrics, but DC currents correspond to singular polarizability for DC conditions, with $\epsilon\sim i\sigma/\omega$, as we saw in for zero-mode oscillators. This will come from $G \underset{\tau\to\infty}{\longrightarrow} \sigma/\epsilon_0$. In any case, we will assume G does not blow up at infinity.

For large ω, χ_e is determined by $G(\tau)$ near $\tau = 0$. Indeed, if

$$G(t) = \sum_n \frac{t^n}{n!} \left.\frac{d^n G}{dt^n}\right|_0 ,$$

$$\begin{aligned}
\chi_e(\omega) &= \sum_n \frac{1}{n!} \left.\frac{d^n G}{dt^n}\right|_0 \int_0^\infty t^n e^{i\omega t} \\
&= \sum_n \left.\frac{d^n G}{dt^n}\right|_0 (-i\omega)^{-(n+1)} = i\frac{G(0)}{\omega} - \frac{G'(0)}{\omega^2} + \dots .
\end{aligned}$$

Note $G(0) = 0$ by continuity from negative τ's, so the leading term is $1/\omega^2$.
With G thus well-behaved,

$$\chi_e(\omega) = \int_0^\infty G(\tau) e^{i\omega\tau}\, d\tau$$

is a well defined integral for all ω with Im $\omega \geq 0$, except for the possible pole at $\omega = 0$. Thus $\chi_e(\omega)$ is an analytic function in the upper half plane.

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Therefore, by Cauchy's theorem, for z in the upper half plane,

$$\chi_e(z) = \frac{1}{2\pi i} \oint_C \frac{\chi_e(\omega')}{\omega' - z} d\omega'$$

with C the contour consisting of the black real axis and the green semicircle, going over 0 and under z if it lies on the real axis. Because $\chi_e(\omega)$ goes to zero at infinity, we can discard the green semicircle. If $z = \omega + i\delta$, with $\delta > 0$

$$\chi_e(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{\chi_e(\omega')}{\omega' - \omega - i\delta}.$$

We are interested in z approaching the real axis from above, $\delta \searrow 0$, so we may use

$$\frac{1}{\omega' - \omega - i\delta} = P\left(\frac{1}{\omega' - \omega}\right) + i\pi\delta(\omega' - \omega),$$

where the principal part P means

$$P\int_{-\infty}^{\infty} d\omega' \frac{1}{\omega' - \omega} f(\omega') := \lim_{\epsilon \to 0} \left(\int_{-\infty}^{\omega - \epsilon} + \int_{\omega + \epsilon}^{\infty} \right) \frac{d\omega'}{\omega' - \omega} f(\omega').$$

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The $\delta(\omega' - \omega)$ term just cancels half the left hand side, so doubling it, for real ω, we have

$$\begin{aligned}
\chi_e(\omega) &= \frac{1}{i\pi} P\int_{-\infty}^{\infty} d\omega' \frac{\chi_e(\omega')}{\omega' - \omega} \\
&= \frac{1}{i\pi} P\int_0^{\infty} d\omega' \left(\frac{\chi_e(\omega')}{\omega' - \omega} - \frac{\chi_e(-\omega')}{\omega' + \omega} \right) \\
&= \frac{1}{i\pi} P\int_0^{\infty} d\omega' \left(\frac{\chi_e(\omega')}{\omega' - \omega} - \frac{\chi_e^*(\omega')}{\omega' + \omega} \right)
\end{aligned}$$

Taking real and imaginary parts separately,

$$\begin{aligned}
\text{Re } \chi_e(\omega) &= \frac{1}{\pi} P\int_0^{\infty} d\omega' \text{Im } \chi_e(\omega') \left(\frac{1}{\omega' - \omega} + \frac{1}{\omega' + \omega} \right) \\
&= \frac{1}{\pi} P\int_0^{\infty} d\omega' \text{Im } \chi_e(\omega') \frac{2\omega'}{\omega'^2 - \omega^2} \\
\text{Im } \chi_e(\omega) &= \frac{-1}{\pi} P\int_0^{\infty} d\omega' \text{Re } \chi_e(\omega') \left(\frac{1}{\omega' - \omega} - \frac{1}{\omega' + \omega} \right) \\
&= \frac{1}{\pi} P\int_0^{\infty} d\omega' \text{Re } \chi_e(\omega') \frac{2\omega}{\omega'^2 - \omega^2}
\end{aligned}$$

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Maxwell's Equations in Linear Media

Recall the basis equations:

$$\begin{aligned}
\vec\nabla \cdot \vec D &= \frac{1}{\epsilon_0}\rho && \text{Gauss for D} \\
\vec\nabla \cdot \vec B &= 0 && \text{Gauss for B} \\
\vec\nabla \times \vec H - \frac{\partial \vec D}{\partial t} &= \mu_0 \vec J && \text{Ampère (+Max)} \\
\vec\nabla \times \vec E + \frac{\partial \vec B}{\partial t} &= 0 && \text{Faraday}
\end{aligned}$$

plus the Lorentz force:

$$\vec F = q(\vec E + \vec v \times \vec B)$$

and the constitutive relations (in frequency space)

$$\vec D = \epsilon \vec E, \qquad \vec B = \mu \vec H.$$

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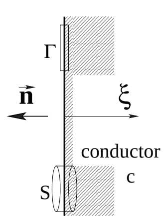
Interface between conductor and non-conductor

Consider the interface between a dielectric and an good conductor c.
Conductor c: if perfect, no $\vec E$. Surface charge Σ and eddy currents can prevent fields from penetrating, so no $\vec H$ inside conductor.
Across the interface:
Faraday on loop $\Gamma \longrightarrow E_\parallel$ continuous
Gauss on pillbox $S \longrightarrow B_\perp$ continuous
Thus just outside the conductor $E_\parallel = 0$, $B_\perp = 0$.



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Good (not perfect) conductor

For good (not perfect) conductor, take $\vec J = \sigma \vec E$ with large *conductivity* σ. Assume time-dependence $\propto e^{-i\omega t}$
Let ξ be distance inside conductor. $\vec H$ varies rapidly with ξ.

$$\vec\nabla \times \vec H_c = \vec J + \frac{\partial \vec D}{\partial t} \approx \sigma \vec E,$$

$$\vec\nabla \times \vec E_c = -\frac{\partial \vec B}{\partial t} = i\omega\mu_c H_c$$

Rapid variation with depth ξ dominates, $\vec\nabla = -\hat n \frac{\partial}{\partial \xi}$, and

$$\vec E_c = \frac{1}{\sigma}\vec J = -\frac{1}{\sigma}\hat n \times \frac{\partial \vec H_c}{\partial \xi}, \qquad \vec H_c = \frac{i}{\omega\mu_c}\hat n \times \frac{\partial \vec E_c}{\partial \xi}$$

$$\vec{E}_c = \frac{1}{\sigma}\vec{J} = -\frac{1}{\sigma}\hat{n}\times\frac{\partial \vec{H}_c}{\partial \xi}, \qquad \vec{H}_c = \frac{i}{\omega\mu_c}\hat{n}\times\frac{\partial \vec{E}_c}{\partial \xi}$$

so $\hat{n}\cdot\vec{H}_c = 0$ and

$$\begin{aligned}\hat{n}\times\vec{H}_c &= \frac{i}{\omega\mu_c}\hat{n}\times\left(\hat{n}\times\frac{\partial \vec{E}_c}{\partial \xi}\right)\\ &= -\frac{i}{\sigma\omega\mu_c}\hat{n}\times\left(\hat{n}\times\left[\hat{n}\times\frac{\partial^2 \vec{H}_c}{\partial \xi^2}\right]\right)\\ &= \frac{i}{\sigma\omega\mu_c}\frac{\partial^2}{\partial \xi^2}\left(\hat{n}\times\vec{H}_c\right).\end{aligned}$$

Simple DEQ, exponential solution, with $\delta = \sqrt{\dfrac{2}{\mu_c\omega\sigma}}$,

$$\vec{H}_c = \vec{H}_\parallel e^{-\xi/\delta}e^{i\xi/\delta},$$

$H_\parallel$ is tangential field outside surface of conductor.

$\vec{E}$ inside conductor and at boundary

From $\vec{H}_c = \vec{H}_\parallel e^{-\xi/\delta}e^{i\xi/\delta}$,

$$\vec{E}_c = -\frac{1}{\sigma}\hat{n}\times\frac{\partial \vec{H}_c}{\partial \xi} = \sqrt{\frac{\mu_c\omega}{2\sigma}}(1-i)\hat{n}\times\vec{H}_\parallel e^{-\xi/\delta}e^{i\xi/\delta},$$

which means, by continuity, that just outside the conductor

$$\vec{E}_\parallel = \sqrt{\frac{\mu_c\omega}{2\sigma}}(1-i)\hat{n}\times\vec{H}_\parallel.$$

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