# Causality

We have seen that the issue of how $\epsilon, \mu$ and $n$ depend on $\omega$ raises questions about causality: Can signals travel faster than $c$, or even backwards in time?
It is very often useful to assume that polarization is linear and local in space, and the polarizability is not time dependent, meaning

$$\vec{D}(\vec{x}, \omega) = \epsilon(\omega) \vec{E}(\vec{x}, \omega),$$

but that does not mean it is local in time, for

$$\begin{aligned}
\vec{D}(\vec{x}, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \vec{D}(\vec{x}, \omega) e^{-i\omega t} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \epsilon(\omega) \vec{E}(\vec{x}, \omega) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \epsilon(\omega) \int_{-\infty}^{\infty} dt' \vec{E}(\vec{x}, t') e^{i\omega t'}
\end{aligned}$$

Let us write $\epsilon(\omega) = \epsilon_0 \left[1 + \chi_e(\omega)\right]$ in terms of the electric susceptibility.

# $G(t-t')$

Let $G(\tau)$ be the fourier transform of $\chi_e(\omega)$,

$$G(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\tau}\chi_e(\omega).$$

Then we have the relation between $\vec{D}$ and $\vec{E}$ given by

$$\vec{D}(\vec{x},t) = \epsilon_0\left\{\vec{E}(\vec{x},t) + \int_{-\infty}^{\infty} d\tau\, G(\tau)\vec{E}(\vec{x},t-\tau)\right\}.$$

Thus we see that $\vec{D}(\vec{x},t)$ depends linearly on the **function** $\vec{E}(\vec{x},t')$ of time $t'$, but not on the single value $\vec{E}(\vec{x},t)$. That is, the dependence is **non-local** in time. Of course if $\epsilon(\omega)$ were constant, $G(\tau)\propto\delta(\tau)$ and we would have the local $\vec{D}(\vec{x},t)=\epsilon\vec{E}(\vec{x},t)$, but that is not the case generally.

As $\vec{D}(\vec{x},t)$ and $\vec{E}(\vec{x},t)$ are both real, $\vec{D}^*(\vec{x},-\omega)=\vec{D}(\vec{x},\omega)$ and similarly for $\vec{E}$, so for real $\omega$, $\epsilon^*(\omega)=\epsilon(-\omega)$. This means $G(\tau)$ is real (for real $\tau$).

# Causality

That the polarization at time $t$ might depend on the electric field at some earlier time $t'$ is not surprising, but shouldn't it be blind to fields at later times? That is, shouldn't we insist $G(\tau) = 0$ for $\tau < 0$)?

Let's consider our oscillator strength model, with

$$\chi_e(\omega) = \frac{\omega_P^2}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

(or a sum of such contributions with different $\omega_0$'s and $\gamma$'s). Then

$$\begin{aligned} G(\tau) &= \frac{\omega_P^2}{2\pi} \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega\tau}}{\omega_0^2 - \omega^2 - i\gamma\omega} \\ &= \frac{\omega_P^2}{4\pi\nu_0} \int_{-\infty}^{\infty} d\omega \, \left( \frac{e^{-i\omega\tau}}{\omega + \nu_0 + i\gamma/2} - \frac{e^{-i\omega\tau}}{\omega - \nu_0 + i\gamma/2} \right). \end{aligned}$$

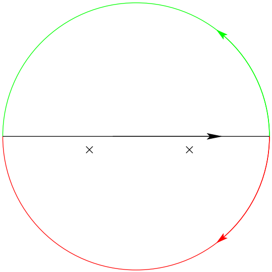
where $\nu_0 = \sqrt{\omega_0^2 - \gamma^2/4}$.

# Evaluating $G(\tau)$ for oscillators

The integrals can be easily done by closing the contours in the complex plane.

The integral

$$\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega\tau}}{\omega \pm \nu_0 + i\gamma/2}$$



can be done by closing the contour with a large semicircle in the upper or lower half plane, whichever gives zero contribution because of the exponential. For negative $\tau$, $|e^{-i\omega\tau}| = e^{-|\tau|\mathrm{Im}\omega}$, so the contribution of the green infinite-radius semicircle in the upper half plane vanishes, and as the integrand is analytic in the enclosed region, the integral is zero. Thus we do have $G(\tau) = 0$ for $\tau < 0$.

For $\tau > 0$, the contour is closed in the lower half plane, and the integral is given by $-2\pi i$ times the sum of the residues, which are $e^{\pm i\nu_0\tau - \gamma\tau/2}$. So the two terms for positive $\tau$ fill in the result:

$$G(\tau) = \omega_P^2 \frac{\sin(\nu_0 \tau)}{\nu_0} \Theta(\tau).$$

Typical values for the lifetime of states, and hence the line-widths of the photons emitted, give $\gamma_i$ from $10^7$/s to $10^9$/s, so the effective $\tau$'s are of the order of nanoseconds. While the response of $\vec{D}$ to $\vec{E}$ is not instantaneous, it is quick.

# Model-independent $\epsilon$'s



We will now find constraints on possible forms of $\epsilon(\omega)$ without assumptions on the model of molecular behavior. We have seen that reality of the fields in the time domain requires $\epsilon^*(\omega) = \epsilon(-\omega)$ for real $\omega$. We have only defined and used $\epsilon(\omega)$ for real $\omega$, but if we would like to continue $\epsilon$ as a complex valued analytic function, we need to extend the constraint to

$$\epsilon^*(\omega^*) = \epsilon(-\omega), \qquad \chi_e^*(\omega^*) = \chi_e(-\omega).$$

We will also insist that $G(\tau)$ is finite and real for positive $\tau$ and zero for negative $\tau$. We might expect $G \underset{\tau\to\infty}{\longrightarrow} 0$ which is true for dielectrics, but DC currents correspond to singular polarizability for DC conditions, with $\epsilon \sim i\sigma/\omega$, as we saw in for zero-mode oscillators. This will come from $G \underset{\tau\to\infty}{\longrightarrow} \sigma/\epsilon_0$. In any case, we will assume $G$ does not blow up at infinity.

For large $\omega$, $\chi_e$ is determined by $G(\tau)$ near $\tau = 0$. Indeed, if

$$G(t) = \sum_n \frac{t^n}{n!} \left.\frac{d^n G}{dt^n}\right|_0,$$

$$\begin{aligned}\chi_e(\omega) &= \sum_n \frac{1}{n!} \left.\frac{d^n G}{dt^n}\right|_0 \int_0^\infty t^n e^{i\omega t} \\ &= \sum_n \left.\frac{d^n G}{dt^n}\right|_0 (-i\omega)^{-(n+1)} = i\frac{G(0)}{\omega} - \frac{G'(0)}{\omega^2} + \ldots .\end{aligned}$$

Note $G(0) = 0$ by continuity from negative $\tau$'s, so the leading term is $1/\omega^2$.
With $G$ thus well-behaved,

$$\chi_e(\omega) = \int_0^\infty G(\tau) e^{i\omega\tau}\, d\tau$$

is a well defined integral for all $\omega$ with Im $\omega \geq 0$, except for the possible pole at $\omega = 0$. Thus $\chi_e(\omega)$ is an analytic function in the upper half plane.

Therefore, by Cauchy's theorem, for $z$ in the upper half plane,

$$\chi_e(z) = \frac{1}{2\pi i} \oint_C \frac{\chi_e(\omega')}{\omega' - z} d\omega'$$

with $C$ the contour consisting of the black real axis and the green semicircle, going over 0 and under $z$ if it lies on the real axis. Because $\chi_e(\omega)$ goes to zero at infinity, we can discard the green semicircle. If $z = \omega + i\delta$, with $\delta > 0$

$$\chi_e(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{\chi_e(\omega')}{\omega' - \omega - i\delta}.$$

We are interested in $z$ approaching the real axis from above, $\delta \searrow 0$, so we may use

$$\frac{1}{\omega' - \omega - i\delta} = P\left(\frac{1}{\omega' - \omega}\right) + i\pi\delta(\omega' - \omega),$$

where the principal part $P$ means

$$P\int_{-\infty}^{\infty} d\omega' \frac{1}{\omega' - \omega} f(\omega') := \lim_{\epsilon \to 0} \left(\int_{-\infty}^{\omega-\epsilon} + \int_{\omega+\epsilon}^{\infty}\right) \frac{d\omega'}{\omega' - \omega} f(\omega').$$

The $\delta(\omega' - \omega)$ term just cancels half the left hand side, so doubling it, for real $\omega$, we have

$$\begin{aligned}
\chi_e(\omega) &= \frac{1}{i\pi} P \int_{-\infty}^{\infty} d\omega' \, \frac{\chi_e(\omega')}{\omega' - \omega} \\
&= \frac{1}{i\pi} P \int_{0}^{\infty} d\omega' \left( \frac{\chi_e(\omega')}{\omega' - \omega} - \frac{\chi_e(-\omega')}{\omega' + \omega} \right) \\
&= \frac{1}{i\pi} P \int_{0}^{\infty} d\omega' \left( \frac{\chi_e(\omega')}{\omega' - \omega} - \frac{\chi_e^*(\omega')}{\omega' + \omega} \right)
\end{aligned}$$

Taking real and imaginary parts separately,

$$\begin{aligned}
\text{Re } \chi_e(\omega) &= \frac{1}{\pi} P \int_{0}^{\infty} d\omega' \text{Im } \chi_e(\omega') \left( \frac{1}{\omega' - \omega} + \frac{1}{\omega' + \omega} \right) \\
&= \frac{1}{\pi} P \int_{0}^{\infty} d\omega' \text{Im } \chi_e(\omega') \frac{2\omega'}{\omega'^2 - \omega^2} \\
\text{Im } \chi_e(\omega) &= \frac{-1}{\pi} P \int_{0}^{\infty} d\omega' \text{Re } \chi_e(\omega') \left( \frac{1}{\omega' - \omega} - \frac{1}{\omega' + \omega} \right) \\
&= \frac{1}{\pi} P \int_{0}^{\infty} d\omega' \text{Re } \chi_e(\omega') \frac{2\omega}{\omega'^2 - \omega^2}
\end{aligned}$$

# Maxwell's Equations in Linear Media

Recall the basis equations:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{D} &= \frac{1}{\epsilon_0}\rho && \text{Gauss for D}\\
\vec{\nabla}\cdot\vec{B} &= 0 && \text{Gauss for B}\\
\vec{\nabla}\times\vec{H}-\frac{\partial\vec{D}}{\partial t} &= \mu_0\vec{J} && \text{Ampère (+Max)}\\
\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t} &= 0 && \text{Faraday}
\end{aligned}$$

plus the Lorentz force:

$$\vec{F}=q(\vec{E}+\vec{v}\times\vec{B})$$

and the constitutive relations (in frequency space)

$$\vec{D}=\epsilon\vec{E},\qquad \vec{B}=\mu\vec{H}.$$

# Interface between conductor and non-conductor



Consider the interface between a dielectric and an good conductor *c*.

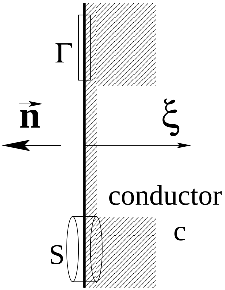
Conductor c: if perfect, no $\vec{E}$. Surface charge $\Sigma$ and eddy currents can prevent fields from penetrating, so no $\vec{H}$ inside conductor.

Across the interface:

Faraday on loop $\Gamma \longrightarrow E_{\parallel}$ continuous

Gauss on pillbox $S \longrightarrow B_{\perp}$ continuous

Thus just outside the conductor $E_{\parallel} = 0$, $B_{\perp} = 0$.



# Good (not perfect) conductor



For good (not perfect) conductor, take $\vec{J} = \sigma\vec{E}$ with large *conductivity* $\sigma$. Assume time-dependence $\propto e^{-i\omega t}$
Let $\xi$ be distance inside conductor. $\vec{H}$ varies rapidly with $\xi$.

$$\vec{\nabla} \times \vec{H}_c = \vec{J} + \frac{\partial \vec{D}}{\partial t} \approx \sigma\vec{E},$$

$$\vec{\nabla} \times \vec{E}_c = -\frac{\partial \vec{B}}{\partial t} = i\omega\mu_c H_c$$

Rapid variation with depth $\xi$ dominates, $\vec{\nabla} = -\hat{n}\frac{\partial}{\partial\xi}$, and

$$\vec{E}_c = \frac{1}{\sigma}\vec{J} = -\frac{1}{\sigma}\hat{n} \times \frac{\partial \vec{H}_c}{\partial \xi}, \qquad \vec{H}_c = \frac{i}{\omega\mu_c}\hat{n} \times \frac{\partial \vec{E}_c}{\partial \xi}$$

$$\vec{E}_c = \frac{1}{\sigma}\vec{J} = -\frac{1}{\sigma}\hat{n} \times \frac{\partial \vec{H}_c}{\partial \xi}, \qquad \vec{H}_c = \frac{i}{\omega\mu_c}\hat{n} \times \frac{\partial \vec{E}_c}{\partial \xi}$$

so $\hat{n} \cdot \vec{H}_c = 0$ and

$$\begin{aligned}
\hat{n} \times \vec{H}_c &= \frac{i}{\omega\mu_c}\hat{n} \times \left(\hat{n} \times \frac{\partial \vec{E}_c}{\partial \xi}\right) \\
&= -\frac{i}{\sigma\omega\mu_c}\hat{n} \times \left(\hat{n} \times \left[\hat{n} \times \frac{\partial^2 \vec{H}_c}{\partial \xi^2}\right]\right) \\
&= \frac{i}{\sigma\omega\mu_c}\frac{\partial^2}{\partial \xi^2}\left(\hat{n} \times \vec{H}_c\right).
\end{aligned}$$

Simple DEQ, exponential solution, with $\delta = \sqrt{\dfrac{2}{\mu_c\omega\sigma}}$,

$$\vec{H}_c = \vec{H}_{\|}e^{-\xi/\delta}e^{i\xi/\delta},$$

$H_{\|}$ is tangential field outside surface of conductor.

# $\vec{E}$ inside conductor and at boundary



From $\vec{H}_c = \vec{H}_\| e^{-\xi/\delta} e^{i\xi/\delta}$,

$$\vec{E}_c = -\frac{1}{\sigma}\hat{n} \times \frac{\partial \vec{H}_c}{\partial \xi} = \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)\hat{n} \times \vec{H}_\| e^{-\xi/\delta} e^{i\xi/\delta},$$

which means, by continuity, that just outside the conductor

$$\vec{E}_\| = \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)\hat{n} \times \vec{H}_\|.$$