Electromagnetic Waves

We begin with waves in a non-conducting uniform linear medium, so we are discussing solutions of Maxwell's equations without sources. As we are assuming no time-dependence of the properties of the medium, we will fourier transform in time and consider the "harmonic" fields, so

$$\begin{array}{lll} \vec{\nabla}\cdot\vec{B}=0 & \vec{\nabla}\times\vec{E}-i\omega\vec{B}=0 & \vec{B}=\mu\vec{H} \\ \vec{\nabla}\cdot\vec{D}=0 & \vec{\nabla}\times\vec{H}+i\omega\vec{D}=0 & \vec{D}=\epsilon\vec{E} \end{array}$$

where the permittivity ϵ and permeability μ are constant in space.

So $$\nabla^2\vec{E}=-\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=-\vec{\nabla}\times(i\omega B)=-i\omega\mu\vec{\nabla}\times\vec{H}$$
$$=-\omega^2\mu\vec{D}=-\omega^2\mu\epsilon\vec{E},$$

which tells us $\left(\nabla^2+\mu\epsilon\omega^2\right)\vec{E}=0$, and by taking the curl of this, the same equation holds for $\vec{B}$.

Plane Waves

If we have a plane wave, $\vec{E}(\vec{x},t) \propto e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, this will satisfy the equation provided the wave number $k := \sqrt{\vec{k}^{\,2}} = \sqrt{\mu\epsilon}\,\omega$. A fixed phase of this wave moves at $\vec{v} = \vec{k}\omega/k^2$ so $v = 1/\sqrt{\mu\epsilon}$, which is called the *phase velocity*. The *index of refraction* is defined as $n = \sqrt{\dfrac{\mu\epsilon}{\mu_0\epsilon_0}}$, and so $v = c/n$.

If we consider plane waves in the x direction, uniform in y and z, we have $u_k(x,t) = ae^{ik(x-vt)} + be^{-ik(x+vt)}$, corresponding to right and left moving sinusoidal waves respectively.

If the medium is nondispersive, so n is constant, we may superimpose these waves with different k to have vaves of arbitrary shape, $u(x,t) = f(x-vt) + g(x+vt)$, but if there is dispersion, having created such a wave packet at $t=0$ will not produce pulses of unchanged shape at later times, because the vt terms in the phase will vary with k.

Thus a general solution of Maxwell's sourceless equations will be a linear superposition of (the real parts of)

$$\left.\begin{aligned}\vec{E}(\vec{x},t) &= \vec{\mathcal{E}} e^{i\vec{k}\cdot\vec{x}-i\omega t}\\ \vec{B}(\vec{x},t) &= \vec{\mathcal{B}} e^{i\vec{k}\cdot\vec{x}-i\omega t}\end{aligned}\right\} \quad \text{with} \quad k^2 = \mu\epsilon\omega^2,$$

but with constraints on $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ coming from the rest of Maxwell's equations.

The divergence equations require $\vec{k}\cdot\vec{\mathcal{E}} = 0$ and $\vec{k}\cdot\vec{\mathcal{B}} = 0$ while one curl equation gives $i\vec{k}\times\vec{\mathcal{E}} = i\omega\vec{\mathcal{B}}$ or $\sqrt{\mu\epsilon}\hat{k}\times\vec{\mathcal{E}} = \vec{\mathcal{B}}$. The magnetic field $\vec{H} = \vec{B}/\mu$ so $\vec{\mathcal{H}} = \hat{k}\times\vec{\mathcal{E}}/Z$ where $Z = \sqrt{\mu/\epsilon}$ is an impediance.

The impediance of free space is $\sqrt{\mu_0/\epsilon_0} = 376.7\,\Omega$.

We have not specified that k and ω are real, which one or the other might not be, as the permittivity and permeability are in general complex. Still, in many contexts they are close to real and if we take $\vec{k}$ to be real, $\mathcal{E}$ and $\mathcal{B}$ will be in phase, with $v\vec{B}$ and $\vec{E}$ equal in magnitude.

Polarization

From these constraints we see that $\vec{\mathcal{E}}$ is a vector perpendicular to the wavenumber $\vec{k}$, and if we set up orthonormal basis vectors $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$ for that plane, with $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$ for that plane, with $\vec{\epsilon}_1 \times \vec{\epsilon}_2 = \hat{k}$, we have $\vec{\mathcal{E}} = E_1\vec{\epsilon}_1 + E_2\vec{\epsilon}_2$, and then $\vec{\mathcal{B}} = \sqrt{\mu\epsilon}(E_1\vec{\epsilon}_2 - E_2\vec{\epsilon}_1)$.

The amplitudes E_1 and E_2 may be complex. If only one of them is nonzero, say E_1, $\vec{E}(\vec{x},t) = \mathrm{Re}\,\left(E_1 e^{i\vec{k}\cdot\vec{x}-i\omega t}\right)\vec{\epsilon}_1 = |E_1|\cos(\vec{k}\cdot\vec{x} - \omega t + \arg E_1)\vec{\epsilon}_1$ so the argument of E_1 is just a phase shift, pretty much irrelevant. In this case, the electric field is *linearly polarized*, oscillating but always in the direction $\vec{\epsilon}_1$. The same is true if E_1 and E_2 are not zero but have the same phase, with the resulting $\vec{E}$ oscillating in the direction $|E_1|\vec{\epsilon}_1 + |E_2|\vec{\epsilon}_2$.

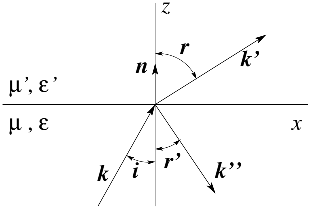
But if $E_2/E_1 = Ae^{i\phi}$ is not real, the electric field components in the two directions are out of phase, and at a given $\vec{x}$ the field sweeps out an ellipse in time.

If $|A| = 1$ and $\phi = \pi/2$, this is a circle, we have the complex $\vec{E}(\vec{x}, t) = E_1(\epsilon_1 + i\epsilon_2)e^{i\vec{k}\cdot\vec{x}-i\omega t}$. The real field $\propto \epsilon_1 \cos \vec{k}\cdot\vec{x} - \epsilon_2 \sin \vec{k}\cdot\vec{x}$ then spirals clockwise as $\vec{x}$ moves along $\vec{k}$, if we are looking into the wave. It spirals counterclockwise as time progresses. This is called a *left circularly polarized* wave, or a wave of *positive helicity*. Of course the opposite phase, with $E_1/E_2 = -i$, is a right circularly polarized wave of negative helicity.

Define $\vec{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\vec{\epsilon}_1 \pm i\vec{\epsilon}_2)$. Then the electric field can be decomposed into $\vec{\epsilon}_\pm$ components rather than $\vec{\epsilon}_j$ $(j = 1, 2)$ components, and each of these is complex. In either case there are four real parameters giving the amplitude of the wave. I think we will not need to discuss the Stokes parameters that give these in terms of measurable quantities.

Reflection and Refraction

Consider a planar interface between two uniform linear media. In each medium, the fields must be a combination of plane waves. Suppose a wave



$$\begin{aligned} \vec{E} &= \vec{E}_0 \, e^{i\vec{k}\cdot\vec{x} - i\omega t}, \\ \vec{B} &= \sqrt{\mu\epsilon}\, \hat{k} \times \vec{E} \end{aligned}$$

($\hat{k}$ is a unit vector) is incident from below, inducing a refracted wave in the upper medium:

$$\vec{E}' = \vec{E}'_0 \, e^{i\vec{k}'\cdot\vec{x} - i\omega t}, \qquad \vec{B}' = \sqrt{\mu'\epsilon'}\, \hat{k}' \times \vec{E}'$$

and a reflected wave in the lower medium

$$\vec{E}'' = \vec{E}''_0 \, e^{i\vec{k}''\cdot\vec{x} - i\omega t}, \qquad \vec{B}'' = \sqrt{\mu\epsilon}\, \hat{k}'' \times \vec{E}''.$$

As all the equations are linear with time-independent parameters, only the one fourier component is involved.

Kinematics

That is, all the waves have the same frequency. This can also be viewed as saying the boundary values must oscillate together in time.

The x and y dependence of the fields at the boundary will also need to match whatever the boundary conditions. This tells us $k_x = k'_x = k''_x$ and $k_y = k'_y = k''_y$. The magnitudes of the three k's are determined by ω and the material parameters, $k = |\vec{k}| = |\vec{k}''| = \omega\sqrt{\mu\epsilon}$, $k' = |\vec{k}'| = \omega\sqrt{\mu'\epsilon'}$. The x, y matching means $k \sin i = k' \sin r = k'' \sin r'$, so $k = k''$ implies $i = r'$, or **the angle of reflection is equal to the angle of incidence.**

But we also have $\dfrac{\sin i}{\sin r} = \dfrac{k'}{k} = \sqrt{\dfrac{\mu'\epsilon'}{\mu\epsilon}} = \dfrac{n'}{n}$, where n and n' are the indices of refraction above and below the interface. This is *Snell's Law*.

The Boundary Conditions

Assume we are considering non-conducting materials with no free charges, so within each material the fields must be differentiable, and at the interface, Gauss' law tells us the normal components of $\vec{B}$ and $\vec{D}$ are continuous, while the other two, integrated on a path just below and just above the interface, tells us the components of $\vec{E}$ and $vecH$ perpendicular to $\vec{n}$ are continuous. Of course below the interface we need to add the incident and reflected fields, so we have

$$\vec{D}\cdot\vec{n} \text{ continuous:} \qquad \left[\epsilon(\vec{E}_0+\vec{E}''_0)-\epsilon'\vec{E}'_0\right]\cdot\vec{n}=0$$

$$\omega\vec{B}\cdot\vec{n} \text{ continuous:} \qquad \left(\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}''_0-\vec{k}'\times\vec{E}'_0\right)\cdot\vec{n}=0$$

$$\vec{E}\times\vec{n} \text{ continuous:} \qquad \left(\vec{E}_0+\vec{E}''_0-\vec{E}'_0\right)\times\vec{n}=0$$

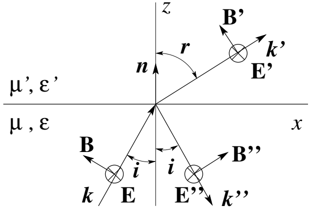
$\vec{H}\times\vec{n}$ continuous:

$$\left[\frac{1}{\mu}\left(\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}''_0\right)-\frac{1}{\mu'}\vec{k}'\times\vec{E}'_0\right]\times\vec{n}=0$$

Plane of Incidence

The transverse conditions on the k's means that they all lie in the *plane of incidence*, defined by the incident direction and the interface normal $\vec{n}$ (assuming $i \neq 0$).
The solution of the interface equations in general is messy, but we can consider separately the two linear polarizations of the incident electric field, in that plane and perpendicular to that plane. Take that plane to be the xz plane.

First consider the incident $\vec{\mathcal{E}}$ into the plane, so $\vec{\mathcal{B}}$ is in the direction shown, and $E_{0\,x} = B_{0\,y} = E_{0\,z} = 0$. Consider a reflection in the plane of incidence, where all E_y, B_x and B_z should change sign, but not the other components, because $\vec{E}$ is a vector and $\vec{B}$ a pseudovector. But this simply reverses the incident fields, and therefore all components must change sign.



$\vec{E}$ perpendicular (continued)

Thus E'_x, E'_z, B'_y, E''_x, E''_z and B''_y must all vanish, and the reflected and transmitted waves are linearly polarized as shown, with all $\vec{E}$'s $\perp$ the plane of incidence.

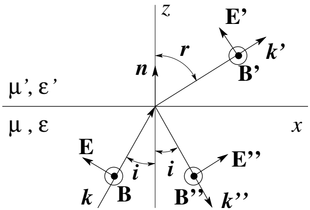
Thus $E_0 + E''_0 - E'_0 = 0$ from the continuity of $\vec{E} \times \vec{n}$, and $\sqrt{\epsilon/\mu}\,\cos i\,(E_0 - E''_0) = \sqrt{\epsilon'/\mu'}\,\cos r\,E'_0$ from the continuity of $\vec{H} \times \vec{n}$. The two equations enable solving for the ratios E'_0/E_0 and E''_0/E_0 whose squares give the transmission and reflection coefficients.

$$\frac{E'_0}{E_0} = \frac{2n\mu'\cos i}{n\mu'\cos i + n'\mu\cos r}, \qquad \frac{E''_0}{E_0} = \frac{n\mu'\cos i - n'\mu\cos r}{n\mu'\cos i + n'\mu\cos r}$$

We see they depend on the ratios n'/n and μ'/μ of the indices of refraction and the permeabilities, but for optical frequencies we may usually take $\mu'/\mu = 1$. The expression is still somewhat complicated. Jackson eliminates the r dependence using $n'\cos r = \sqrt{n'^2 - n'^2\sin^2 r} = \sqrt{n'^2 - n^2\sin^2 i}$.

$\vec{E}$ in the plane of incidence

The argument for the opposite linear polarization, with $\vec{E}$ in the plane of incidence and $\vec{B}$ perpendicular to it proceeds similarly. The $\vec{E} \times \vec{n}$ equation gives $(E - E'') \cos i = E' \cos r$



and the $\vec{H} \times \vec{n}$ equation gives us

$$\frac{1}{\mu}(E_0 + E_0'') = \frac{1}{\mu'} E_0' \frac{k'}{k} = \frac{1}{\mu'} E_0' \frac{n'}{n}$$

Again the results are not simple. We get

$$\begin{aligned} \frac{E_0'}{E_0} &= \frac{2n\mu' \cos i}{n'\mu \cos i + n\mu' \cos r} \\ \frac{E_0''}{E_0} &= \frac{n'\mu \cos i - n\mu' \cos r}{n'\mu \cos i + n\mu' \cos r} \end{aligned}$$

Normal Incidence

If the angle of incidence goes to zero, the two results must converge, as the plane of incidence is not well defined. As $i = r = 0$ in that limit, the $\vec{E}_\perp$ results give

$$\frac{E_0'}{E_0} = \frac{2}{1 + \dfrac{n'}{n}\dfrac{\mu}{\mu'}}, \qquad \frac{E_0''}{E_0} = 1 - 2\frac{1}{1 + \dfrac{n}{n'}\dfrac{\mu'}{\mu}},$$

and so does the $\vec{E}$ in the plane, except that the sign of E_0''/E_0 is reversed, but that is because the directions defining E_0'' are reversed in the two cases.

Simplification if $\mu = \mu'$

We are often interested in visible light in dielectric materials which have little magnetic susceptibility, so let us assume $\mu = \mu'$. Then for $\vec{E}_\perp$,

$$\frac{E_0'}{E_0} = \frac{2n\cos i}{n\cos i + n'\cos r} = \frac{2\cos i \sin r}{\sin(i+r)}$$

$$\frac{E_0''}{E_0} = \frac{n\cos i - n'\cos r}{n\cos i + n'\cos r} = \frac{\sin(r-i)}{\sin(r+i)}.$$

and for $\vec{E}_\parallel$,

$$\frac{E_0'}{E_0} = \frac{2n\cos i}{n'\cos i + n\cos r} = \frac{4\sin r\cos i}{\sin(2i)+\sin(2r)}$$

$$\frac{E_0''}{E_0} = \frac{\sin(2i)-\sin(2r)}{\sin(2i)+\sin(2r)} = \frac{\tan(i-r)}{\tan(i+r)}.$$

Normal incidence, again, with $\mu = \mu'$

For normal incidence, we need to not use the sines, which vanish. Using the $\vec{E}_\perp$ definition where positive values are all in the same direction, we see

$$\frac{E_0'}{E_0} = \frac{2n}{n+n'}, \qquad \frac{E_0''}{E_0} = \frac{n-n'}{n+n'}.$$

Note that if the reflection is off a more dense medium ($n' > n$), the sign of $\vec{E}$ is reversed on reflection, which is the origin of the rule we use in elementary discussions of interference, that there is an extra phase shift by π under reflection off a more-dense medium.

Brewster angle

For the electric field in the plane of incidence, there is an angle for which E''_0 vanishes, called *Brewster's angle*. Assuming $\mu' = \mu$ for simplicity (and it is nearly true for optics in dielectric materials), this is when $n' \cos i = n \cos r$, and as $n \sin i = n' \sin r$, multiplying them, we see $\sin 2i = \sin 2r$, or $r = \pi/2 - i$, and the reflected and refracted waves are perpendicular to each other. Then $n' = n \sin i / \sin r = n \tan i$, so

$$\text{Brewster's angle:} \qquad i_B = \tan^{-1}\left(\frac{n'}{n}\right).$$

This is a way to get complete linear polarization of light. It also indicates that even at other angles, the transmission of one polarization will be greater than the other.

Total Internal Reflection

Earlier we derived Snell's law, $n \sin i = n' \sin r$, which shows that, if $n > n'$, for an angle of incidence i greater than $\sin^{-1}(n'/n)$ we need a refraction angle with a sine greater than 1. How do we interpret that? Really we got this by requiring $k_x = k'_x$ with $k' = n'k/n$, from which $k'_z = \sqrt{k'^2 - k_x^2}$ which is imaginary (say $i\kappa$ under these circumstances. Then the fields

$$\vec{E}\,' = \vec{E}\,'_0 e^{ik_x x - \kappa z - i\omega t}, \quad \vec{H}\,' = \frac{1}{\mu' \omega} \vec{k}' \times \vec{E}\,'.$$

Note the fields fall off exponentially with depth z into the less dense medium for angles beyond that of total internal reflection, so there is no continuing beam of refracted light, though there is field close to the interface.

There is no flow of energy into the n' medium, because the Poynting vector's z component is

$$\begin{aligned}\vec{S}\cdot\vec{n} &= \frac{1}{2}\mathrm{Re}\,\left[\vec{n}\cdot\left(\vec{E}'\times\vec{H}'^{*}\right)\right] \\ &= \frac{1}{2\omega\mu'}\mathrm{Re}\,\vec{n}\cdot\left[\vec{E}'\times\left(\vec{k}'\times\vec{E}'^{*}\right)\right] \\ &= \frac{1}{2\omega\mu'}\mathrm{Re}\,\vec{n}\cdot\vec{k}'\left|\vec{E}'_0\right|^2 = 0,\end{aligned}$$

as $\vec{n}\cdot\vec{k}'$ is pure imaginary.

The naive geometrical picture of well defined beams that are reflected exactly at the boundary should be doubted. Perhaps it is effectively off a plane somewhat inside the boundary, and the reflected ray would intersect the incident ray inside the second medium. This is called the *Goos-Hänchen effect*. To really describe how much the reflected beam is moved from the naive path, we would need wave packets in x rather than pure wavenumber. We are not going to pursue this.