Physics 504, Spring 2011
Electricity and Magnetism

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Course Information

- Instructor:
  - Joel Shapiro
  - Serin 325
  - 5-5500 X 3886, shapiro@physics
- Book: Jackson: Classical Electrodynamics (3rd Ed.)
- Web home page: `www.physics.rutgers.edu/grad/504` contains general info, syllabus, lecture and other notes, homework assignments, *etc.*
- Classes: ARC 207, Monday and Thursday, 10:20 (sharp!) - 11:40
- Homework: there will be one or two projects, and homework assignments every week or so. Due dates to be discussed.
- Exams: a midterm and a final.
- Office Hour: Tuesdays, 3:30–4:30, in Serin 325, or by arrangement.

Course Content

Last term you covered Jackson, Chapters 1—6.3
We will cover a large fraction of the rest of Jackson, but we will have to leave out quite a bit.

Everything comes from Maxwell's Equations and the Lorentz Force. We will discuss:

- EM fields in matter, and at boundaries
- EM fields confined: waveguides, cavities, optical fibers
- Sources of fields: antennas and their radiation, scattering and diffraction
- Relativity, and relativistic formalism for E&M
- Relativistic particles
- other gauge theories (maybe)

Maxwell's Equations

At a fundamental level, electromagnetic fields obey Maxwell's equations with sources given by all charges and currents:

$$\begin{aligned}
\vec\nabla\cdot\vec E &= \frac{1}{\epsilon_0}\rho_{\rm all} && \text{Gauss for E}\\
\vec\nabla\cdot\vec B &= 0 && \text{Gauss for B}\\
\vec\nabla\times\vec B-\frac{1}{c^2}\frac{\partial\vec E}{\partial t} &= \mu_0\vec J_{\rm all} && \text{Ampère (+Max)}\\
\vec\nabla\times\vec E+\frac{\partial\vec B}{\partial t} &= 0 && \text{Faraday}
\end{aligned}$$

and charged particles experience the Lorentz force:

$$\vec F = q\left(\vec E+\vec v\times\vec B\right)$$

Potentials

As $\vec B$ has zero divergence[1] ($\vec\nabla\cdot\vec B=0$), there are vector fields $\vec A(\vec r)$ for which $\vec B(\vec r)=\vec\nabla\times\vec A(\vec r)$. Then

$$\frac{\partial\vec B}{\partial t}=\vec\nabla\times\frac{\partial\vec A}{\partial t},$$

so Faraday's law tells us

$$\vec\nabla\times\left(\vec E+\frac{\partial\vec A}{\partial t}\right)=0,$$

and the term in parenthesis is[2] the gradient of some function, $-\Phi$. Then

$$\vec E=-\vec\nabla\Phi-\frac{\partial\vec A}{\partial t}.$$

[1] $\vec B$ corresponds to a closed 2-form, which in Euclidean space means it is exact, $\mathbf{B}=d\mathbf{A}$ for some 1-form $\mathbf{A}$.
[2] So the 1-form $\mathbf{E}+\partial\mathbf{A}/\partial t$ is closed, and exact, and equal to $d(-\Phi)$.

Gauge Invariance

Because only the curl of $\vec A$ is determined by $\vec B$, $\vec A$ is determined only up to a divergence[3]. Though such an ambiguity will affect $\vec E$, this can be compensated for in our choice of Φ. Thus the electromagnetic fields are unchanged by a gauge transformation:

$$\begin{aligned}
\vec A &\to \vec A+\vec\nabla\Lambda,\\
\Phi &\to \Phi-\frac{\partial\Lambda}{\partial t}.
\end{aligned}$$

Because of gauge invariance, the physical equations (Maxwell) do not fully determine $\vec A$ and Φ, but we may *impose* a **gauge condition**. Two popular choices are:

$$\begin{aligned}
\vec\nabla\cdot\vec A+\frac{1}{c^2}\frac{\partial\Phi}{\partial t} &= 0 && \text{Lorenz gauge,}\\
\vec\nabla\cdot\vec A &= 0 && \text{Coulomb gauge.}
\end{aligned}$$

[3] a closed or exact form

Solving the Wave Equation

Either gauge choice is possible because a Λ to adjust $\vec{A}$ can be found by Green function for the wave or Laplace equation.
With the Lorenz gauge, Maxwell's equations give

$$\begin{aligned}\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} &= -\frac{\rho}{\epsilon_0}\\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\mu_0\vec{J}\end{aligned}$$

Both of these are wave equations with specified source:

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi f(\vec{x},t),$$

with some form of boundary condition. If there are no boundaries, solution by Fourier transform is best.

Fourier transforms

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Let's review Fourier transforms.
Given an integrable real or complex-valued function on the real line $\Re$, $f:\Re\to\mathbb{C}$, we define its Fourier transform $\tilde{f}:\Re\to\mathbb{C}$ by

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt\, f(t)e^{i\omega t}.$$

This can be inverted, because

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\tilde{f}(\omega)e^{-i\omega t} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\int_{-\infty}^{\infty} dt'\,f(t')e^{i\omega(t'-t)}\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt'\,f(t')\int_{-\infty}^{\infty} d\omega\,e^{i\omega(t'-t)}\\ &= \int_{-\infty}^{\infty} dt'\,f(t')\delta(t'-t) = f(t).\end{aligned}$$

$$\text{so}\qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\tilde{f}(\omega)e^{-i\omega t}.$$

Fourier transforms convert derivatives to algebra

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If $g(t) = \dfrac{d}{dt}f(t)$,

$$\begin{aligned}\tilde{g}(\omega) &= \int_{-\infty}^{\infty} dt\,e^{i\omega t}\frac{d}{dt}f(t)\\ &= \int_{-\infty}^{\infty} dt\,e^{i\omega t}\frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega'\,\tilde{f}(\omega')e^{-i\omega' t}\right)\\ &= -i\int_{-\infty}^{\infty} d\omega'\,\omega'\,\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\,e^{i(\omega-\omega')t}\\ &= -i\omega\tilde{f}(\omega).\end{aligned}$$

Confusion and Inconsistencies

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There are lots of arbitrary choices made here, and not made consistently by different authors.

Which is the fourier transform, $f(t)\to\tilde{f}(\omega)$ or the reverse, and which is the reverse transform?

Which one gets the minus sign in the exponential? In fact, everyone chooses $\mathsf{exp}(i\omega t)$ for $f(t)\to\tilde{f}(\omega)$ but $\mathsf{exp}(-ikx)$ for $f(x)\to\tilde{f}(k)$ for spatial functions.

Which transform gets the $1/2\pi$ factor?
$\tilde{f}\to f$ does here, in §6.4
or do they each get a $\sqrt{1/2\pi}$, as on p. 69,

or should we transform to frequency ν instead of angular frequency ω, with $e^{2\pi i\nu t}$, in which case we don't need any factors of 2π in front.
That's what Wikipedia does, but no physics book I know of does.

Back to solving the wave equation

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So if we fourier transform in time, our wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = -4\pi f(\vec{x},t)$$

transforms to

$$\left(\nabla^2 + k^2\right)\tilde{\Psi}(\vec{x},\omega) = 4\pi\tilde{f}(\vec{x},\omega),$$

with $k := \omega/c$. For a fixed ω this is now just a three-dimensional differential equation, depending on $\vec{x}$. It is the *inhomogeneous Helmholtz* equation. It can be solved by finding the suitable Green function:

$$G_k(\vec{x},\vec{x}\,') \ni \left(\nabla^2 + k^2\right)G_k(\vec{x},\vec{x}\,') = -4\pi\delta^3(\vec{x}-\vec{x}\,').$$

In general we might be interested in the Green function with some specified spatial boundary conditions. Let us first consider, however, an unrestricted region, for which the Green function will be translationally invariant, $G_k(\vec{x},\vec{x}\,') = G_k(\vec{x}-\vec{x}\,')$, and rotationally symmetric, $G_k(\vec{x},\vec{x}\,') = G_k(|\vec{x}-\vec{x}\,'|)$.

Free space Helmholtz Green function

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In that case, with $R = |\vec{x}-\vec{x}\,'|$, we have[4]

$$\frac{1}{R}\frac{d^2}{dR^2}\left(RG_k(R)\right) + k^2G_k(R) = 0 \quad \text{for } R > 0.$$

Thus $G_k(R) = AG^{(+)}(R) + BG^{(-)}(R)$, with

$$G_k^{(\pm)}(R) = \frac{e^{\pm ikR}}{R}.$$

As $\vec{\nabla}\cdot\vec{\nabla}G_k(\vec{r}) = -k^2G_k(\vec{r}) - 4\pi\delta^3(\vec{r})$, if we integrate inside a small sphere of radius ϵ and use Gauss' law:

$$\int_{|r|\le\epsilon}\vec{\nabla}\cdot\vec{\nabla}G_k(\vec{r}) = \int_{|r|=\epsilon}\hat{e}_r\cdot\vec{\nabla}G_k(\vec{r}).$$

[4] In spherical coordinates, $\nabla^2\Psi$ is given by Jackson's equation 3.1, which on a function with no angular dependence reduces to $r^{-1}d^2(r\Phi)/dr^2$. We will derive 3.1 and generalizations of other coordinate systems in a few weeks.

On the small sphere, the radial component of the gradient of $G_k(\vec r)$ is

$$\hat e_r \cdot \vec\nabla G_k(\vec r) = \frac{-1+ik\epsilon}{\epsilon^2} A e^{ik\epsilon} + \frac{-1-ik\epsilon}{\epsilon^2} B e^{-ik\epsilon} \underset{\epsilon\to 0}{\longrightarrow} -\frac{A+B}{\epsilon^2}$$

so the surface integral is $-4\pi(A+B)$.
On the other hand, when integrating $-k^2 G_k(\vec r) - 4\pi\delta^3(\vec r)$, over the small sphere, the first term vanishes as the volume integral ($\propto r^3$) dominates the Green function ($\propto 1/r$), and the integral of $\delta^3(\vec r) = 1$. Thus

$$A + B = 1.$$

Again, $G_k^{(\pm)}(\vec r) = \dfrac{e^{\pm ik|\vec r|}}{|\vec r|}$. and $AG^{(+)} + (1-A)G^{(-)}$ is a Green function for the Helmholtz equation.

Time Dependent Green Function

Green functions depend on boundary conditions. We left ambiguity in behavior for $t \to \pm\infty$, so have arbitrariness in A, B.
The Green function for wave equation in space-time:

$$\left(\nabla^2_{\vec x} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G^{(\pm)}(\vec x, t, \vec x\,', t') = -4\pi\delta^3(\vec x - \vec x\,')\delta(t - t')$$

can be found by fourier transforming ($t \to \omega$) the source into $-4\pi\delta^3(\vec x - \vec x\,')e^{i\omega t'}$, so the fourier transform of the Green function is $G_k^{(\pm)}(\vec x - \vec x\,')e^{i\omega t'} = G_k^{(\pm)}(R)e^{i\omega t'}$, where $R = |\vec x - \vec x\,'|$. Taking the inverse transform ($\omega \to t$) we have

$$\begin{aligned} G^{(\pm)}(\vec x, t, \vec x\,', t') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, G^{(\pm)}_{\omega/c}(R)\, e^{-i\omega(t-t')} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{\pm i\omega R/c}}{R} e^{-i\omega\tau} \end{aligned}$$

where $\tau = t - t'$.

Note, as we might expect, that it depends only on the difference of the two points in spacetime (on $\vec x - \vec x\,'$ and $t - t'$). Also note the integral over ω gives a simple delta function,

$$\begin{aligned} G^{(\pm)}(\vec x, t, \vec x\,', t') &= G^{(\pm)}(R, \tau) = \frac{1}{R}\delta\left(\tau \mp \frac{R}{c}\right) \\ &= \frac{\delta\left(t' - \left[t \mp |\vec x - \vec x\,'|/c\right]\right)}{|\vec x - \vec x\,'|}. \end{aligned}$$

The delta function requires $t' = t \mp R/c$ to contribute, and R/c is always nonnegative, so for $G^{(+)}$ only $t' \le t$ contibutes, or sources only affect the wave function after they act. Thus $G^{(+)}$ is called a *retarded* Green function, as the affects are retarded (after) their causes. On the other hand, $G^{(-)}$ is the *advanced* Green function, giving effects which preceed their causes.

In and Out Fields

If the sources $f(\vec x, t)$ are only nonzero in a finite time interval $[t_i, t_f]$, any solution of the wave equation $\Psi(\vec x, t)$ must obey the homogeneous wave equation for $t < t_i$ and for $t > t_f$. As that equation is deterministic, if we define $\Psi_{\rm in}(\vec x, t)$ and $\Psi_{\rm out}(\vec x, t)$ to be solutions, for all t, of the homogeneous equation which agree with $\Psi(\vec x, t)$ for $t < t_i$ and for $t > t_f$ respectively, we must have

$$\begin{aligned} \Psi(\vec x, t) &= \Psi_{\rm in}(\vec x, t) + \int d^3x' \int_{-\infty}^{\infty} dt'\, G^{(+)}(\vec x - \vec x\,', t - t') f(\vec x\,', t') \\ &= \Psi_{\rm out}(\vec x, t) + \int d^3x' \int_{-\infty}^{\infty} dt'\, G^{(-)}(\vec x - \vec x\,', t - t') f(\vec x\,', t'). \end{aligned}$$

The solution $\Psi(\vec x, t)$ is not determined by the sources alone, because we could have an arbitrary free wave coming from $t = -\infty$, but we may view the difference $\Psi_{\rm out} - \Psi_{\rm in}$ as the field produced by the sources after they have acted. If the sources are considered fixed, independent of the field, this will not depend on the incident wave, but if the sources are there because of the incident wave, we may view this difference as the scattered wave.

We will skip Jackson §6.5

In "Ponderable Media"

Up to now, we have talked fundamental fields interacting with all charges. Restricting our focus to distances large compared to 10^{-14} m ~ 10 fm, matter consists of point charge nuclei Ze and electrons with charge $-e$. On an atomic scale we would have wildly fluctuating fields dependent on the detailed positions on innumerable atoms. For many purposes we are not interested in the detailed fields, which we will now call $\vec e$ and $\vec b$, but in their averages over regions large compared to an atom, say $R = 10^{-8}$ m. With a smearing function such as

$$f(\vec x\,') = \frac{1}{\pi^{3/2}R^3} e^{|\vec x\,'|^2/R^2}, \quad \text{which gives} \int f(\vec x) d^3x = 1,$$

we define *macroscopic* fields

$$\begin{aligned} \vec E(\vec x, t) &= \langle \vec e(\vec x, t)\rangle := \int d^3x' f(\vec x\,')\vec e(\vec x - \vec x\,', t), \\ \vec B(\vec x, t) &= \langle \vec b(\vec x, t)\rangle := \int d^3x' f(\vec x\,')\vec b(\vec x - \vec x\,', t). \end{aligned}$$

These averaged fields are known as the electric field and **magnetic induction**. The latter is an unfortunate historical necessity, as the magnetic field will be defined differently in a moment.
The homogeneous Maxwell equations are linear, so the averaging passes right through, and we have

$$\vec{\nabla}\cdot\vec{B}=0, \qquad \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0.$$

The averaging of the charges, however, is more interesting. We divide the charges into some we consider free, with a charge density $\eta_{\rm free}$, and some we consider to be there in response to the fields, with a charge density we call $\eta_{\rm bound}$, because it is usually identified as charges bound to the molecules.

The free charges can be described by their positions $\vec{x}_j$, with $\eta_{\rm free}(\vec{x})=\sum_j q_j\delta^3(\vec{x}-\vec{x}_j)$. The bound charges are better described in terms of the center of mass of the molecule $\vec{x}_n$ and the charge's displacement from it, $\vec{x}_{jn}$ with $\vec{x}_j=\vec{x}_n+\vec{x}_{jn}$.

The bound density $\eta_n(\vec{x},t)$ for one molecule n is a sum over its bound charges $j(n)$, so
$\eta_{\rm bound}(\vec{x},t)=\sum_n\eta_n(\vec{x},t)=\sum_n\sum_{j(n)}q_j\delta(\vec{x}-\vec{x}_n-\vec{x}_{jn})$,
and the smeared density for molecule n is

$$\begin{aligned}\langle\eta_n(\vec{x},t)\rangle &= \sum_{j(n)}q_j\int d^3x'f(\vec{x}\,')\delta(\vec{x}-\vec{x}\,'-\vec{x}_n-\vec{x}_{jn})\\ &= \sum_{j(n)}q_jf(\vec{x}-\vec{x}_n-\vec{x}_{jn})\\ &\approx \sum_{j(n)}q_j\Big(f(\vec{x}-\vec{x}_n)-\sum_\alpha x_{jn\,\alpha}\frac{\partial}{\partial x_\alpha}f(\vec{x}-\vec{x}_n)\\ &\qquad+\frac{1}{2}\sum_{\alpha\beta}x_{jn\,\alpha}\,x_{jn\,\beta}\,\frac{\partial^2}{\partial x_\alpha\partial x_\beta}f(\vec{x}-\vec{x}_n)+\ldots\Big).\end{aligned}$$

The displacements of the bound charges $\vec{x}_{jn}$ from their molecule's center is small compared to the averaging scale R, so we may expand f in a power series, which separates out the x_{jn} dependence from the $\vec{x}-\vec{x}_n$.

Multipole Moments

We define the multipole moments for the molecule:

$$q_n=\sum_{j(n)}q_j,\qquad \vec{p}_n=\sum_{j(n)}q_j\vec{x}_{jn},$$

$$\mathbf{Q}_{n\,\alpha\beta}=3\sum_{j(n)}q_j\,x_{jn\,\alpha}\,x_{jn\,\beta}$$

are the monopole (or charge), dipole and quadripole moments of the molecule about the center of mass.

Aside: Jackson uses Q', presumably because this includes a monopole monopole moment $3\sum q_jx_{jn}^2$ in addition to the true quadripole part.

The averaged charge density is then equivalent to a density of charges, dipoles and quadripoles given by the density of molecules. Let us define a *macroscopic charge density*

$$\rho(\vec{x},t)=\left\langle\eta_{\rm free}+\sum_n q_n\delta(\vec{x}-\vec{x}_n)\right\rangle,$$

where the $\sum_n$ is over the molecules, and considers the net charge of any molecule n as located at its center of mass. Performing the smearing for the bound charges,

$$\rho(x,t)=\langle\eta_{\rm free}\rangle+\sum_n q_nf(\vec{x}-\vec{x}_n).$$

This gives the first term in $\langle\eta_n(\vec{x},t)\rangle$.

Polarization

Define the *macroscopic polarization*

$$\begin{aligned}\vec{P}(\vec{x},t) &= \left\langle\sum_n\vec{p}_n\delta(\vec{x}-\vec{x}_n)\right\rangle\\ &= \int d^3x'f(\vec{x}-\vec{x}\,')\sum_n\vec{p}_n\delta(\vec{x}\,'-\vec{x}_n)\\ &= \sum_n\vec{p}_nf(\vec{x}-\vec{x}_n)=\sum_{n,j}\vec{q}_j\vec{x}_{jn}f(\vec{x}-\vec{x}_n)\end{aligned}$$

which is the smeared out dipole moment of the molecules. Then

$$\vec{\nabla}\cdot\vec{P}(\vec{x},t)=\sum_{n,j,\alpha}\vec{q}_j\,x_{jn\,\alpha}\,\frac{\partial}{\partial x_\alpha}f(\vec{x}-\vec{x}_n).$$

This gives the second term in $\langle\eta_n(\vec{x},t)\rangle$.

Quadripole Moment

Next define the *macroscopic quadripole density*[5]

$$\begin{aligned}\mathbf{Q}_{\alpha\beta}(\vec{x},t) &= \frac{1}{6}\left\langle\sum_n\mathbf{Q}_{n\,\alpha\beta}\delta(\vec{x}-\vec{x}_n)\right\rangle\\ &= \frac{1}{6}\int d^3x'f(\vec{x}-\vec{x}\,')\sum_n\mathbf{Q}_{n\,\alpha\beta}\,\delta(\vec{x}\,'-\vec{x}_n)\\ &= \frac{1}{6}\sum_n\mathbf{Q}_{n\,\alpha\beta}f(\vec{x}-\vec{x}_n)=\frac{1}{2}\sum_{n,j}q_jx_{jn\,\alpha}x_{jn\,\beta}f(\vec{x}-\vec{x}_n)\end{aligned}$$

Then $\displaystyle\sum_{\alpha\beta}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}\mathbf{Q}_{\alpha\beta}(\vec{x},t)$

$$=\frac{1}{2}\sum_{n,j}q_j\,x_{jn\,\alpha}\,x_{jn\,\beta}\,\frac{\partial^2}{\partial x_\alpha\partial x_\beta}f(\vec{x}-\vec{x}_n).$$

This gives the third term in $\langle\eta_n(\vec{x},t)\rangle$.

[5] Caution: note the apparent inconsistency by a factor of 6 between $\mathbf{Q}$ and $\mathbf{Q}_n$

Maxwell's Gauss' Law

We see comparing these to our first expansion that

$$\langle \eta(\vec{x},t)\rangle = \rho(\vec{x},t) - \vec{\nabla}\cdot\vec{P}(\vec{x},t) + \sum_{\alpha\beta}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}\mathbf{Q}(\vec{x},t).$$

Then upon smearing Gauss' law for the microscopic electric field $\vec{e}(\vec{x},t)$, $\epsilon_0\vec{\nabla}\cdot\vec{E}(\vec{x},t) = \langle\eta(\vec{x},t)\rangle$, gives us

$$\rho(\vec{x},t) = \epsilon_0\vec{\nabla}\cdot\vec{E}(\vec{x},t) + \vec{\nabla}\cdot\vec{P}(\vec{x},t) - \sum_{\alpha\beta}\frac{\partial^2}{\partial x_\alpha\partial x_\beta}\mathbf{Q}_{\alpha\beta}.$$

$$\text{Let}\quad \vec{D}_\alpha(\vec{x},t) = \epsilon_0\vec{E}_\alpha(\vec{x},t) + \vec{P}_\alpha(\vec{x},t) - \sum_\beta\frac{\partial}{\partial x_\beta}\mathbf{Q}_{\alpha\beta},$$

so that

$$\vec{\nabla}\cdot\vec{D}(\vec{x},t) = \rho(\vec{x},t).$$

This is the macroscopic Gauss' Law for $\vec{D}$.