Lectures 25-26 April 29, May 4 2010

Electromagnetism controls most of physics from the atomic to the planetary scale, and we have spent nearly a year exploring the concrete consequences of Maxwell's equations and the Lorentz force.

But electromagnetism holds one clue to set our imagination free to go further.
In the gauge invariance of the vector potential, it points to the elegant structure of

LOCAL SYMMETRY

a generalization of a global symmetry of electromagnetism which is not obvious classically.

Topics we need to introduce

We first need to discuss several topics from outside the realm of classical electromagnetism:

- Lattice approach to field theories
- Internal field variables
- Global internal symmetries
- Quantum mechanics of a charged particle
- How to make global symmetries local

This will give us

- minimal substitution
- covariant derivatives
- electromagnetic field strengths as a form of curvature
- non-Abelian gauge theories (QCD, GSW, standard model)

Lattice approach to field theories

Field theories may be approached as the continuum limit of discrete degrees of freedom on a lattice.

Matter fields have their degrees of freedom defined only at sites $\vec{x} = (an_x, an_y, an_z)$, with integers n_i and lattice spacing a.

For relativistic treatments we will also discretize time.

What are the degrees of freedom ϕ_j at each site j? ϕ_j lives in some domain: reals? complex numbers? a finite set (like for spin 1/2 Ising model)? a real vector?

Spins on a Lattice

Consider having at each site a spin $\vec{S}_j$ of fixed length but of variable direction, with an interaction $H = J \sum_{\text{nn}} \vec{S}_i \cdot \vec{S}_j$ between nearest-neighbor lattice sites i and j. Without any coupling to spatial things (such as $(\vec{r}_i - \vec{r}_j) \cdot (\vec{S}_j - \vec{S}_i)$, spins are in separate space, not necessarily 3D.

Invariant under rotations by an orthogonal matrix $\mathbf{R}$, $\vec{S}_j \to \mathbf{R}\vec{S}_j$, each term $\vec{S}_i \cdot \vec{S}_j$ in the Hamiltonian in unchanged.

This holds also in the continuum limit, with $\vec{S}(\vec{x})$ a field taking independent (though not uncorrelated) values at each spatial point.

Here the nearest-neighbor coupling becomes $\sum_{\mu\alpha} \partial_\mu S_\alpha \partial^\mu S_\alpha$. That keeps our internal space and real space uncoupled.

If the internal space is $\mathbb{R}^3$, then we might also have $(\vec{\nabla} \cdot \vec{S})^2$ terms, and then space and spin would be coupled with an $\vec{L} \cdot \vec{S}$ term.

Global internal symmetries

If there is no coupling of the spins with space, the symmetry under rotation in spin space is called an *internal symmetry*.

Isotopic spin. $\Psi := \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$. Strong interactions are invariant under SU(2) transformations

$$\Psi(x^\mu) \to e^{i\vec{\omega}\cdot\vec{I}}\Psi(x^\mu),$$

$\left(\vec{I}\right)_i$ are pauli spin matrices σ_i acting on the doublet Ψ.

The isospin rotation has to be the same at all points in spacetime for this to be a symmetry. Thus it is a *global* symmetry.

Quantum mechanics of a charged particle

In Quantum Mechanics, $\vec{p} \to -i\hbar\vec{\nabla}$ and $E \to i\hbar\partial/\partial t$. Then a free nonrelativistic particle with $E = \vec{p}^2/2m$ is described by a wave function

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$$

which is Schrödinger's equation for a free particle.

Notice ψ must be complex.
If $\psi(x^\mu)$ is a solution, so is $e^{i\lambda}\psi(x^\mu)$, as long as λ is a constant. Global symmetry under phase change.

Ms. Noether tells us: $\exists$ conserved current. For our non-relativistic particle: $\vec{J} = \rho\vec{v} \to -i\hbar q\psi^*\vec{\nabla}\psi$.

Conserved current

More generally, the phase induces $\Delta\phi = i\lambda\phi$, the current

$$J^\mu = -\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\Delta\phi$$

is just like the kinetic energy part of the lagrangian density with one derivative left out.

For example, the Dirac Lagrangian $\mathcal{L} = i\hbar\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$ gives a conserved current $J^\mu = \bar{\psi}\gamma^\mu\psi$,
while for the Klein-Gordon lagrangian for a charged scalar field, $\mathcal{L} = \hbar^2(\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi^*\phi$ gives
$J_\mu = \frac{i\hbar}{2m}\left(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right)$.

This conserved current is a consequence of the *global* symmetry.

Minimal Substitution

If the particle is charged and in the presence of an external field $A^\mu(x^\nu)$, this interaction can be incorporated by "minimal substitution" into the free particle lagrangian, which is to say that $\vec{p} \to \vec{p} - q\vec{A}/c$, $E \to E - q\Phi$, so for a non-relativistic particle

$$i\hbar \frac{\partial \psi}{\partial t} = \left(q\Phi - \frac{\hbar^2}{2m} \left(\vec{\nabla} + iq\vec{A}/\hbar c \right)^2 \right) \psi.$$

$q\Phi$ is the potential energy. Might recognize the $\vec{A}$ term from Schnetzer's derivation of Hamiltonian.
Important point: Not invariant under $A_\mu \to A_\mu - \partial_\mu \Lambda$. Only effect is on derivative — adds a piece $iq\partial_\mu \Lambda / c$ to each derivative operator, or $\partial_\mu \to e^{-iq\Lambda/c} \partial_\mu e^{iq\Lambda/c}$.
So if ψ satisfies the equation in the original gauge, $\psi' = e^{-iq\Lambda/c}\psi$ saitisfies the equation in the transformed gauge. So we have a symmetry under a more involved gauge transformation.

Gauge invariance with matter

Conclusion:

$$
\begin{aligned}
A_\mu &\rightarrow A_\mu + \partial_\mu \Lambda \\
&\qquad (\Phi \rightarrow \Phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}, \qquad \vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda) \\
\psi &\rightarrow e^{-iq\Lambda/c}\psi
\end{aligned}
$$

is an invariance of the theory, and is the correct form of a gauge transformation.

Here we have the required local phase change, or rotation, as a result of demanding gauge invariance, but we now ask the reverse — can we turn a global symmetry of the matter fields into a local symmetry by having gauge fields with a gauge invariance.

Non-Abelian Symmetry

Let us develop the idea of a locally invariant gauge theory more formally, starting with matter fields with a symmetry group.

Let the matter fields be a set of N real fields $\phi_i(x^\mu)$ with an internal symmetry group $\mathcal{G}$ which acts with representation M, that is, $G \in \mathcal{G}$ acts on the ϕ fields by

$$G : \phi_i(x) \mapsto \phi_i'(x) = \sum_j M_{ij}(G)\phi_j(x). \tag{1}$$

To be a symmetry the Lagrangian must be invariant. The usual field theory kinetic term is $\frac{1}{2}\sum_{\mu,i}\left(\partial_\mu \phi_i\right)\left(\partial^\mu \phi_i\right)$, invariant if M is a constant orthogonal matrix $\sum_k M_{ki}M_{kj} = \delta_{ij}$. Any potential term of the forms $V(\sum_i \phi_i^2)$ is also invariant. Individual components of ϕ have no intrinsic meaning, as M mixes them up.

V might not be invariant under all of O(N) but only under a subgroup. Example: SU(3) of color acts on triplet of complex quark fields, 6 real components, kinetic energy is invariant under O(6)∼SU(4), but full lagrangian is invariant only under an SU(3) subgroup.

So we have a symmetry group $\mathcal{G}$ which has generators L_b which form a basis of the "Lie algebra" $\mathfrak{G}$.
There are as many independent L_b's as the dimension of the group or algebra. For example, 3 for rotations in 3D or SU(2), 8 for SU(3), $N(N-1)/2$ for SO(N), rotations in N dimensions.

Discretization

On our lattice, the degrees of freedom $\phi_i(\mathbf{x})$ are replaced by $\phi_i(\vec{n})$ for $\vec{n} \in \mathbb{Z}^4$ with $\mathbf{x}^\mu = an^\mu$. The mass term

$$-\frac{m^2}{2}\int d^4x \sum_i \phi_i^2(\mathbf{x}) \rightarrow -\frac{m^2a^4}{2}\sum_{\vec{n}\in\mathbb{Z}^4}\sum_i \phi_i^2(\vec{n}).$$

Similarly for any other potential (single site) term.

Kinetic energy: replace derivative by finite difference. The simplest substitution is to replace

$$\partial_\mu\phi_i(\mathbf{x}) \rightarrow \frac{1}{a}\left(\phi_i(\vec{n}+\vec{\Delta}_\mu) - \phi_i(\vec{n})\right),$$

where $\vec{\Delta}_\mu$ is 1 in the μ direction and 0 in the others. Note the best $x^\nu = an^\nu + \frac{1}{2}a\delta^\nu_\mu$, the center of the link between the two sites. $(\Delta\phi)^2$ gives mass-like terms and nearest-neighbor terms $\sum_i \phi_i(\vec{n}+\vec{\Delta}_\mu)\phi_i(\vec{n})$. Each is invariant under rotations if the same rotation is used at all sites, so we have a global symmetry.

In a relativistic field theory, all information is local. Why should symmetry at one site know about a distant one? Could we have a **local** symmetry?

Mass terms and potential terms, which depend only on a single site, are okay because they don't know the difference between local and global.
But the nearest neighbor term

$$\sum_i \phi_i(\vec{n}+\vec{\Delta}_\mu)\phi_i(\vec{n})$$
$$\rightarrow M_{ik}(G(\vec{n}+\vec{\Delta}_\mu))M_{ij}(G(\vec{n}))\,\phi_k(\vec{n}+\vec{\Delta}_\mu)\phi_j(\vec{n})$$
$$= \Big(M^{-1}(G(\vec{n}+\vec{\Delta}_\mu))M(G(\vec{n}))\Big)_{kj}\;\phi_k(\vec{n}+\vec{\Delta}_\mu)\phi_j(\vec{n})$$

is not invariant because

$$M^{-1}(G(\vec{n}+\vec{\Delta}_\mu))M(G(\vec{n})) \neq 1$$

if the G's (and hence the M's) vary from point to point.

Parallel Transport

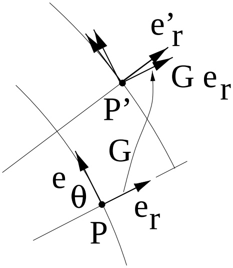
The problem, of course, is that we need a measure of how much ϕ changes from point to point which does not depend on the arbitrary basis vectors used at each point to describe ϕ in terms of components. We need a method of looking at $\phi(\vec{x})$ as if it were at $\vec{x}\,'$, so we can subtract it from $\phi(\vec{x}\,')$ to find the change. That is, we need a definition of parallel transport — moving the object (say a vector) ϕ from one point to another without rotating or otherwise distorting it.

We considered this problem in Lecture 4 when we considered the gradient operator on vectors in curvilinear coordinates. Let's recall what we did then.

Rule for parallel transport

As an example, it might help to think of an ordinary vector in the plane, expressed in polar coordinates. Consider the unit basis vectors $\vec{e}_r$ and $\vec{e}_\theta$ at the point P.

If we transport $\vec{e}_r$ to the point P' while keeping it parallel to what it was, we arrive at the vector labelled Ge_r, which is not the same as the unit radial vector e'_r at the point P'.



Note that if we have a vector $\vec{V}\,' = V'_r \vec{e}\,'_r + V'_\theta \vec{e}\,'_\theta$ at P' which is unchanged (parallel transported) from the vector $\vec{V} = V_r \vec{e}_r + V_\theta \vec{e}_\theta$ at P, we **do not have** $V'_r = V_r$.

Here we had an *a priori* rule for parallel transport. But if we allow gauge invariance, the rule becomes a dynamical variable, a new degree of freedom.

For each pair of nearest neighbors, that is, for each link, we need a group element which specifies how to parallel-transport the basis vectors.
Thus we have a lot of new degrees of freedom, but we do have a local symmetry.

This gives us a **gauge field theory**.

The matter fields in a gauge theory transform according to some representation of the group, but the gauge fields, which define the parallel transport, take values in the group itself, or in the Lie algebra formed from the generators of the group.

So for example, for colored quarks, the quarks transform as a three-complex-dimension representation of the SU(3) group, but there are 8 gluons, because there are 8 generators of SU(3).

Covariant Derivative

We have seen that each link will have a group element G which defines parallel transport, so that transporting $\phi_i(\vec{n})$ to the neighboring site $\vec{n} + \vec{\Delta}_\mu$ gives $\sum_j M_{ij}(G)\phi_j(\vec{n})$, and so the change in ϕ which might enter our lagrangian is

$$\Delta\phi_i = \phi_i(\vec{n} + \vec{\Delta}_\mu) - \sum_j M_{ij}(G)\phi_j(\vec{n}).$$

In the continuum limit the fields change little from one site to the next, so

$$\phi_i(\vec{n} + \vec{\Delta}_\mu) \approx \phi_i(\vec{n}) + a\partial_\mu\phi_i.$$

and parallel transport should not require more than a small rotations, proportional to the lattice spacing, so we can write[1]

$$G = e^{iag\mathcal{A}}, \quad M(G) = M(e^{iag\mathcal{A}}) \approx 1 + iagM(\mathcal{A}).$$

[1] The scale of generators $\mathcal{A}$ is somewhat arbitrary — we include a factor of g, which will be called the fundamental charge, here, but some people do not.

Here the generator $\mathcal{A} \in \mathfrak{G}$, the Lie algebra generating $\mathcal{G}$. So to $o(a^1)$,

$$\Delta\phi_i = a \left(\partial_\mu \phi_i - ig \sum_j M_{ij}(\mathcal{A})\phi_j \right) .$$

In the continuum limit we define $1/a$ times this to be *the covariant derivative.*

We need a covariant derivative for each link, of which there are four emerging from each site. So we have a 4-vector of Lie-algebra valued fields $\mathcal{A}_\mu$, which have Lie-algebra components multiplying the basis generators L_b, so $\mathcal{A}_\mu(x^\nu) = \sum_b A_\mu^{(b)}(x^\nu) L_b$. Then the covariant derivative, acting on the matter field, is

$$\begin{aligned} \left(D_\mu \phi\right)_j &= \partial_\mu \phi_j - ig \sum_{kb} A_\mu^{(b)} M_{jk}(L_b)\phi_k \\ D_\mu \phi &= \partial_\mu \phi - ig A_\mu^{(b)} M(L_b)\phi. \end{aligned}$$

Gauge Transformations

On the lattice the terms we were just discussing are

$$\phi(\vec{n} + \vec{\Delta}_\mu) \cdot M(G_L) \cdot \phi(\vec{n}),$$

where G_L is the group transformation associated with the link $(\vec{n}, \vec{n} + \vec{\Delta}_\mu)$ that implements a parallel transport. What happens to G_L under local group transformations?

Think of gauge transformations passively — that is, $\phi(\vec{n})$ doesn't change but its description in terms of components $\phi_i(\vec{n})$ does. Parallel transport across a link doesn't change either, but its group element does.

If G_L transports ϕ_p to ϕ_q at site q with $M(G_L) \cdot \phi(x_p)$ in our original set of basis vectors, and we do a gauge transformation $\phi_p \to \phi'_p = M(G_p) \cdot \phi_p$,
$\phi_q \to \phi'_q = M(G_q) \cdot \phi_q$, the parallel transport in the new basis is

$$\phi'_q = M(G_q) \cdot M(G_L) \cdot M(G_p)^{-1} \cdot \phi_p = M(G_q G_L G_p^{-1}) \cdot \phi_p,$$

so the gauge field which does parallel transport $p \to q$ is $G_q G_L G_p^{-1}$

Non-Abelian Gauge transform

Now our gauge transform acts on both matter fields and gauge fields:

$$\Lambda : \begin{cases} \phi(x_p) \to M(G_p) \cdot \phi(x_p) \\ \phi(x_q) \to M(G_q) \cdot \phi(x_q) \\ G_L \to G_q G_L G_p^{-1} \end{cases}$$

This gauge transformation is a **local symmetry** of the gauge field theory. Let's verify that this is an invariance of the nearest neighbor term:

$$\begin{aligned} &\phi(x_q) \cdot M(G_L) \cdot \phi(x_p) \\ &\quad = \phi_i(x_q) M_{ij}(G_L) \phi_j(x_p) \\ &\quad\to M_{ik}(G_q)\phi_k(x_q) M_{ij}(G_q G_L G_p^{-1}) M_{j\ell}(G_p)\phi_\ell(x_p) \\ &\qquad = \phi_k(x_q) M^{-1}_{ki}(G_q) M_{ij}(G_q G_L G_p^{-1}) M_{j\ell}(G_p)\phi_\ell(x_p) \\ &\qquad = \phi_k(x_q) M_{k\ell}(G_L)\phi_\ell(x_p) = \phi(x_q) \cdot M(G_L) \cdot \phi(x_p), \end{aligned}$$

where we have used the orthogonality of $M(G_q)$ and the fact that the M's are a representation, and therefore $M^{-1}_{ki}(G_q) M_{ij}(G_q G_L G_p^{-1}) M_{j\ell}(G_p) = M_{k\ell}(G_L)$.

In the Continuum

In the continuum limit, $G_L = e^{iag\mathcal{A}_L}$,
$M(G_L) = M(e^{iag\mathcal{A}_L}) \approx 1 + iagM(\mathcal{A}_\mathcal{L})$. The gauge transform group element Λ is not necessarily a small change, but we can still write it as the exponential of a generator, $\Lambda(x^\mu) = e^{i\lambda(x^\mu)}$ with $\lambda(x^\mu)$ a Lie algebra element, $\lambda(x^\mu) = \sum_b \lambda^{(b)}(x^\mu) L_b$. We may assume that $\lambda^{(b)}$, though not small, changes slowly (o(a)) across a link. So if the gauge transformation takes $\phi(\mathbf{x}) \to \phi'(\mathbf{x})$ and $A_\mu^{(b)}(\mathbf{x}) \to A_\mu'^{(b)}(\mathbf{x})$, we have

$$\begin{aligned}
\phi'(\mathbf{x}) &= e^{i\sum_b \lambda^{(b)}(\mathbf{x})M(L_b)}\phi(\mathbf{x}), \\
e^{iagA_\mu'^{(b)}(\mathbf{x})} &= e^{i\lambda(\mathbf{x}+\frac{1}{2}a\Delta_\mu)}\, e^{iagA_\mu^{(b)}(\mathbf{x})L_b}\, e^{-i\lambda(\mathbf{x}-\frac{1}{2}a\Delta_\mu)}.
\end{aligned}$$

We placed x^μ in the middle of a link.

To first order in a

In expanding in powers of a, we need care that $\lambda(\mathbf{x})$, $\partial_\mu\lambda(\mathbf{x})$, and $\mathcal{A}_\mu$ do not commute, so we expand Λ rather than λ,

$$\begin{aligned} e^{iag\mathcal{A}_\mu} &\rightarrow 1 + iag\mathcal{A}_\mu, \\ e^{i\lambda(\mathbf{x}\pm\frac{1}{2}a\Delta\mu)} &\rightarrow e^{i\lambda(\mathbf{x})} \pm \frac{1}{2}a\partial_\mu\left[e^{i\lambda(\mathbf{x})}\right], \end{aligned}$$

and plugging these in, we get

$$\begin{aligned} 1+iag\mathcal{A}'_\mu &= \left(e^{i\lambda}+\frac{1}{2}a\partial_\mu e^{i\lambda}\right)(1+iag\mathcal{A}_\mu)\left(e^{-i\lambda}-\frac{1}{2}a\partial_\mu e^{-i\lambda}\right) \\ &= 1 + iage^{i\lambda}\mathcal{A}_\mu e^{-i\lambda} \\ &\qquad +\frac{1}{2}a\left(\partial_\mu e^{i\lambda}\right)e^{-i\lambda} - \frac{1}{2}ae^{i\lambda}\left(\partial_\mu e^{-i\lambda}\right) \end{aligned}$$

Note from $\partial_\mu\left(e^{i\lambda}e^{-i\lambda}\right) = 0$ that the third and fourth terms are equal, so we can drop the third and double the fourth.

Gauge transform of Gauge Field

This gives

$$\begin{aligned} \mathcal{A}'_\mu &= e^{i\lambda}\mathcal{A}_\mu e^{-i\lambda} + \frac{i}{g} e^{i\lambda}\partial_\mu e^{-i\lambda} \\ &= e^{i\lambda}\left(\mathcal{A}_\mu + \frac{i}{g}\partial_\mu\right) e^{-i\lambda} \end{aligned}$$

Electromagnetism?

Gauge transformations for charged wavefunctions were just phase shifts, rotations in two dimensions. There is only one generator L_1 with $G = e^{i\lambda L_1}$, so everything commutes. The gauge transformation takes

$$\phi \to \phi' = \begin{pmatrix} \mathrm{Re}\,\Phi' \\ \mathrm{Im}\,\Phi' \end{pmatrix} = \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} \mathrm{Re}\,\Phi \\ \mathrm{Im}\,\Phi \end{pmatrix}$$

or $\Phi' = e^{i\lambda}\Phi$, and

$$A'_\mu = e^{i\lambda}\left(A_\mu + \frac{i}{g}\partial_\mu\right) e^{-i\lambda} = A_\mu + g^{-1}\partial_\mu\lambda.$$

Gauge transform in E&M

So for electromagnetism,

$$\begin{aligned} \Phi' &= e^{i\lambda}\Phi \\ A'_\mu &= A_\mu + g^{-1}\partial_\mu\lambda. \end{aligned}$$

is what we are used to. But this simplicity only holds for an **Abelian** group, where all generators commute.

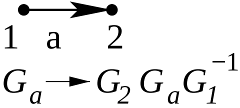
More generally, we need

$$\begin{aligned} \phi'(x^\nu) &= e^{iM(\lambda(x^\nu))}\,\phi(x^\nu) \\ \mathcal{A}'_\mu(x^\nu) &= e^{i\lambda(x^\nu)}\mathcal{A}_\mu(x^\nu)e^{-i\lambda(x^\nu)} + \frac{i}{g}e^{i\lambda(x^\nu)}\partial_\mu e^{-i\lambda(x^\nu)} \end{aligned}$$

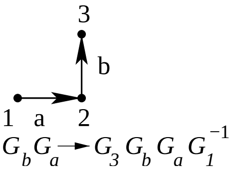
Pure Gauge Terms in $\mathcal{L}$

We have learned how to formulate the interaction of matter fields with the gauge fields, both on the lattice and in the continuum. But what about the pure gauge field part of $\mathcal{L}$?

Can't depend on just one link a and be gauge invariant, because choosing a gauge transform $G_1 = G_a$ makes $G'_a = 1\!\!1$, nothing to depend on.



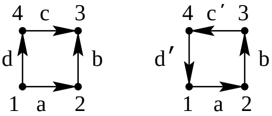
Simplest way to get rid of the gauge dependance of $G_a = e^{iag\mathcal{A}_x(\mathbf{x}_a)}$ on $G_2 = e^{i\lambda(\mathbf{x}_2)}$ is to premultiply it by G_b,



$$\begin{aligned} G_b G_a \quad &\to \quad G_3 G_b G_2^{-1} G_2 G_a G_1^{-1} \\ &= G_3 G_b G_a G_1^{-1}. \end{aligned}$$

Independent of G_2, but ...

There is still a gauge dependence on the endpoints of the path, however, so the best thing to do is close the path. To do so, we are traversing some links backwards from the way they were defined, but from that definition in terms of parallel transport it is clear that the group element associated with taking a link backwards is the inverse of the element taken going forwards. So



the group element associated with the closed path on the right (which is called a plaquette) is $G_P = G_d^{-1} G_c^{-1} G_b G_a$, which transforms under gauge transformations as

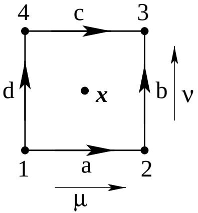
$$\begin{aligned} G_P \to G'_P &= \left(G_4 G_d G_1^{-1}\right)^{-1} \left(G_3 G_c G_4^{-1}\right)^{-1} \\ &\qquad \left(G_3 G_b G_2^{-1}\right) \left(G_2 G_a G_1^{-1}\right) \\ &= G_1 G_d^{-1} G_4^{-1} G_4 G_c^{-1} G_3^{-1} G_3 G_b G_2^{-1} G_2 G_a G_1^{-1} \\ &= G_1 G_d^{-1} G_c^{-1} G_b G_a G_1^{-1} \\ &= G_1 G_P G_1^{-1}. \end{aligned}$$

So the plaquette group element is not invariant.

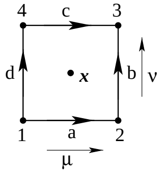
Invariant function on placquette

$G_P \to G_1 G_p G_1^{-1}$ is not invariant, but is simpler. In the continuum limit each G_L is small transformation, and G_c differs from G_a proportional to the lattice spacing, so G_P close to the identity, $G_P - 1\!\!1 \approx$ a generator in the Lie algebra. The Killing form acting on that generator will provide us with an invariant. Let us define

$\mathcal{F}_{\mu\nu} = -ia^{-2}g^{-1}(G_P - 1)$ to be the field-strength tensor, where μ and ν are the directions of links a and b respectively. Let us take $\mathbf{x}$ in the center of the placquette.



Expanding each link to order $\mathcal{O}(a^2)$



$$G_a \approx 1+iag\mathcal{A}_\mu(\mathbf{x}-\frac{1}{2}a\Delta_\nu)-\frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x}-\frac{1}{2}a\Delta_\nu)$$
$$\approx 1+iag\mathcal{A}_\mu(\mathbf{x})-\frac{1}{2}ia^2g\partial_\nu\mathcal{A}_\mu(\mathbf{x})-\frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x})$$

$$G_c^{-1} \approx 1-iag\mathcal{A}_\mu(\mathbf{x}+\frac{1}{2}a\Delta_\nu)-\frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x}+\frac{1}{2}a\Delta_\nu)$$
$$\approx 1-iag\mathcal{A}_\mu(\mathbf{x})-\frac{1}{2}ia^2g\partial_\nu\mathcal{A}_\mu(\mathbf{x})-\frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x}),$$

so, to order a^2,

$$\begin{aligned} G_P = & \left(1 - iag\mathcal{A}_\nu(\mathbf{x}) + \frac{1}{2}ia^2g\partial_\mu\mathcal{A}_\nu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\nu^2(\mathbf{x})\right) \\ & \left(1 - iag\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}ia^2g\partial_\nu\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x})\right) \\ & \left(1 + iag\mathcal{A}_\nu(\mathbf{x}) + \frac{1}{2}ia^2g\partial_\mu\mathcal{A}_\nu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\nu^2(\mathbf{x})\right) \\ & \left(1 + iag\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}ia^2g\partial_\nu\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x})\right) \\ = & \; 1 + a^2g\left\{g\left[\mathcal{A}_\mu(\mathbf{x}),\mathcal{A}_\nu(\mathbf{x})\right] + i\partial_\mu\mathcal{A}_\nu(\mathbf{x}) - i\partial_\nu\mathcal{A}_\mu(\mathbf{x})\right\} \end{aligned}$$

Evaluation of $\mathcal{F}_{\mu\nu}$

Thus

$$\mathcal{F}_{\mu\nu}(\mathbf{x}) = \partial_\mu \mathcal{A}_\nu(\mathbf{x}) - \partial_\nu \mathcal{A}_\mu(\mathbf{x}) - ig \left[\mathcal{A}_\mu(\mathbf{x}), \mathcal{A}_\nu(\mathbf{x})\right].$$

Note that $\mathcal{F}_{\mu\nu}$ is

- a Lie-algebra valued field, $\mathcal{F}_{\mu\nu}(\mathbf{x}) = \sum_b F^{(b)}_{\mu\nu}(\mathbf{x}) L_b$.
- An antisymmetric tensor, $\mathcal{F}_{\mu\nu}(\mathbf{x}) = -\mathcal{F}_{\nu\mu}(\mathbf{x})$.
- Because the Lie algebra is defined in terms of the structure constants, ${c_{ab}}^d$ by

  $$[L_a, L_b] = i{c_{ab}}^d L_d,$$

  the field-strength tensor may also be written

  $$F^{(d)}_{\mu\nu} = \partial_\mu A^{(d)}_\nu - \partial_\nu A^{(d)}_\mu + g{c_{ab}}^d A^{(a)}_\mu A^{(b)}_\nu.$$

Before we turn to the Lagrangian, let me point out a crucial relationship between the covariant derivatives and the field-strength. If we take the commutator of covariant derivatives

$$D_\mu = \partial_\mu - igA_\mu^{(b)} L_b$$

at the same point but in different directions,

$$\begin{aligned}
[D_\mu, D_\nu] &= [\partial_\mu - ig\mathcal{A}_\mu, \partial_\nu - ig\mathcal{A}_\nu] \\
&= -ig\partial_\mu \mathcal{A}_\nu - g^2 \mathcal{A}_\mu \mathcal{A}_\nu - (\mu \leftrightarrow \nu) \\
&= -g^2 [\mathcal{A}_\mu, \mathcal{A}_\nu] - ig\partial_\mu \mathcal{A}_\nu + ig\partial_\nu \mathcal{A}_\mu \\
&= -ig\mathcal{F}_{\mu\nu}.
\end{aligned}$$

Notice that although the covariant derivative is in part a differential operator, the commutator has neither first or second derivatives left over to act on whatever appears to the right. It does need to be interpreted, however, as specifying a representation matrix that will act on whatever is to the right.

The Lagrangian Density for $\mathcal{F}$

We have seen that the field strength $\mathcal{F}_{\mu\nu}$ is an element of the Lie algebra which transforms by conjugation, $\mathcal{F}_{\mu\nu} \to G\mathcal{F}_{\mu\nu}G^{-1}$ under gauge transformation G at one vertex.

For the Lagrangian we need an invariant quadratic function on the algebra — fortunately that is just the Killing form, $\beta(L_a, L_b) = 2\delta_{ab}$ if we have normalized our generators L_a in the usual way. This is familiar in the form " L^2 is invariant" for the rotations. Thus we may take

$$\mathcal{L} = -\frac{1}{4}\sum_b F^{(b)}_{\mu\nu} F^{(b)\,\mu\nu}$$

to be the pure gauge term. Obviously that agrees with our E&M lagrangian, where there is only one term in the sum over b.

Theory of Almost Everything

All together, if we have a non-Abelian gauge (Yang-Mills) field interacting with Dirac particles that transform under a representation t^b (that is, $t^b = D(L_b)$ in the representation for the fermions), together with some scalar particles ϕ transforming under a possibly different representation $\bar{t}$, the langrangian density is

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4}F^{(b)}_{\mu\nu}F^{(b)\,\mu\nu} + i\bar{\psi}\gamma^{\mu}\left(\partial_\mu - igA^{(b)}_{\mu}t^b\right)\psi \\ & +\frac{1}{2}\left[\left(\partial_\mu - igA^{(b)}_{\mu}\bar{t}^b\right)\phi\right]^T\left[\left(\partial^\mu - igA^{(b)\,\mu}\bar{t}^b\right)\phi\right].\end{aligned}$$

Now you know everything about the non-Abelian theories that run the universe (except for gravity).