Lecture 24 April 26, 2010

Charges in motion produce fields.
Fields affect the motion of charges.
Often we think of one causing the other.
But of course this is mutual interaction, complicated.

Why can we so often ignore the back reaction? Consider non-relativistic particle radiating, $P = 2e^2a^2/3c^3$. Over time interval T, if $\vec{a}$ roughly constant, $\Delta v \approx aT$, typical energy $\sim \frac{1}{2}mv^2 \sim ma^2T^2$, so radiated energy small if

$$\frac{2e^2a^2T}{3c^3} \ll ma^2T^2 \Longrightarrow T \gg \tau \quad := \quad \frac{2e^2}{3mc^3} \left(\frac{e^2}{6\pi\epsilon_0 mc^3} \text{ in SI units} \right).$$

Usually true! For electron, $\tau = \dfrac{2e^2}{3m_ec^3} = 6.26 \times 10^{-24}$ s, $c\tau = 1.88$ femtometers.

Another example: For particle in circular motion,

$$E_{\text{rad}} \sim \frac{2e^2}{c^3}\omega^4 r^2 \times \frac{2\pi}{\omega} \quad \text{compared to} \quad E_0 \sim m\omega^2 r^2,$$

so the radiated power has only an adiabatic effect provided $\omega\tau \ll 1$.

But over time, adiabatic effects are significant. Let $\vec{F}_{\text{ext}}$ be the force ignoring radiative reaction, so without radiation $m\dot{\vec{v}} = \vec{F}_{\text{ext}}$, but power is lost at $P(t) = 2e^2(\dot{\vec{v}})^2/3c^3$, exerting a damping force $\vec{F}_{\text{rad}}$ which does negative work on it:

$$\begin{aligned} -\int_{t_1}^{t_2} P(t)\,dt &= \int_{t_1}^{t_2} \vec{F}_{\text{rad}}\vec{v}(t)\,dt = -\frac{2e^2}{3c^3}\int_{t_1}^{t_2} \dot{\vec{v}} \cdot \dot{\vec{v}}\,dt \\ &= \frac{2e^2}{3c^3}\int_{t_1}^{t_2} \ddot{\vec{v}} \cdot \vec{v}\,dt - \frac{2e^2}{3c^3}\dot{\vec{v}} \cdot \vec{v}\Big|_{t_1}^{t_2}. \end{aligned}$$

Excuses for dropping endpoint term: quasi-periodic motion, or pick times for which $\dot{\vec{v}} \cdot \vec{v} = 0$, or bounded motion over a long time.

Abraham-Lorentz Equation

Then we have

$$\vec{F}_{\text{rad}} = \frac{2e^2}{3c^3}\ddot{\vec{v}} = m\tau\ddot{\vec{v}}.$$

This gives the **Abraham-Lorentz equation** of motion, involving $\ddot{v}$, that is, third time derivatives of position, or *jerk*. This violates usual mechanics rules, for good reason. Particles can take off $x(t) = x_0 e^{t/\tau}$ without any external forces! But as perturbations on reasonable motion, this is useful.

To allow for the damping but not positive feedback, take $\vec{F}_{\text{rad}} = \frac{2e^2}{3c^3}\frac{d}{dt}F_{\text{ext}}/m = \tau dF_{\text{ext}}/dt$, and evaluate the stream derivative for F_{ext}:

$$m\dot{\vec{v}} = F_{\text{ext}} + \tau\left[\frac{\partial F_{\text{ext}}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)F_{\text{ext}}(\vec{x},t)\right]. \qquad (1)$$

Example: charge bound by spherical potential

Consider charge with *potential energy* $U(r)$. In absence would have conserved energy E and a conserved angular momentum $\vec{L}$, a force $\vec{F}_{\text{ext}} = -\frac{dU}{dr}\frac{\vec{r}}{r}$ and an acceleration $\dot{\vec{v}} = -\frac{1}{m}\frac{\vec{r}}{r}\frac{dU}{dr}$. The particle loses energy at a rate $P(t)$, so

$$\frac{dE}{dt} = -\frac{2e^2}{3c^3}(\dot{\vec{v}})^2 = -\frac{2e^2}{3m^2c^3}\left(\frac{dU}{dr}\right)^2 = -\frac{\tau}{m}\left(\frac{dU}{dr}\right)^2.$$

The rate of change of the angular momentum is

$$\frac{d\vec{L}}{dt} = \vec{r}\times m\dot{\vec{v}} = \vec{r}\times\left[-\frac{\vec{r}}{r}\frac{dU}{dr} - \tau(\vec{v}\cdot\vec{\nabla})\left(\frac{\vec{r}}{r}\frac{dU}{dr}\right)\right].$$

The first term contains $\vec{r}\times\vec{r}$ and vanishes. So does the term where the gradient acts on dU/dr, as that also contains $\vec{r}\times\vec{r}$.

Using

$$\vec{v} \cdot \vec{\nabla} \hat{e}_r = \vec{v}/r - \vec{r}(\vec{r} \cdot \vec{v})/r^2,$$

we have

$$\frac{d\vec{L}}{dt} = -\tau \vec{r} \times \left(\frac{\vec{v}}{r} - \vec{r} \frac{\vec{r} \cdot \vec{v}}{r^2} \right) \frac{dU}{dr} = \frac{-\tau}{m} \vec{L} \frac{1}{r} \frac{dU}{dr}.$$

As we expect that the damping terms have a small effect over one almost-closed orbit, we can consider the averages over an orbit,

$$\begin{aligned} \left\langle \frac{dE}{dt} \right\rangle &= -\frac{\tau}{m} \left\langle \left(\frac{dU}{dr} \right)^2 \right\rangle, \qquad (2) \\ \left\langle \frac{d\vec{L}}{dt} \right\rangle &= -\frac{\tau}{m} \vec{L} \left\langle \frac{1}{r} \frac{dU}{dr} \right\rangle. \end{aligned}$$

In one orbit the damping can be ignored, we can calculate $\left\langle \left(\frac{dU}{dr}\right)^2 \right\rangle$ and $\left\langle \frac{1}{r}\frac{dU}{dr} \right\rangle$ from E and $\vec{L}$ ignoring damping, and treat (2) as ordinary differential equations. An electron in the lowest Bohr orbit of hydrogen would spiral into the nucleus after a time

$$t = \frac{a_0^3}{9c^3\tau^2} = 15 \ \text{ picoseconds!}$$

Of course quantum mechanics forbids transitions to nonexistent or occupied energy levels, but this treatment does give good results for $\ell \to \ell - 1$ transition rates, according to Bohr's correspondence principle. (This is a homework I might have assigned you, but you are lazy ☺).

Line Width

Consider instead a 1-D harmonic oscillator, $k = m\omega_0^2$, $F_{\text{ext}} = -m\omega_0^2 x$. Assuming $\omega_0\tau \ll 1$, we can use

$$m\dot{\vec{v}} = F_{\text{ext}} + \tau\left[\frac{\partial F_{\text{ext}}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)F_{\text{ext}}(\vec{x},t)\right] = -m\omega_0^2(x+\tau v)$$

or

$$m\ddot{x} + m\omega_0^2\tau\dot{x} + m\omega_0^2 x = 0,$$

a simple ODE with solutions $x(t) = x_0 e^{-\alpha t}$ with $\alpha^2 - \tau\omega_0^2\alpha + \omega_0^2 = 0$, or

$$\alpha = \frac{1}{2}\tau\omega_0^2 \pm i\omega_0\sqrt{1-(\tau\omega_0/2)^2} \approx \frac{1}{2}\tau\omega_0^2 \pm i(\omega_0 - \tau^2\omega_0^3/8).$$

The real part of this is the decay constant $\Gamma/2$, while the imaginary part is the angular frequency, slightly shifted by the damping,

$$\omega = \omega_0 + \Delta\omega, \qquad \text{with } \Delta\omega = -\tau^2\omega_0^3/8.$$

Thus if the oscillator is set going at time zero, it will emit radiation proportional to $(\ddot{x}(t))^2$, which will not have a pure frequency, but rather the distribution of frequencies of the amplitude of emitted radiation is

$$E(\omega) \propto \int_0^\infty e^{-\alpha t} e^{i\omega t} dt = \frac{1}{\alpha - i\omega},$$

whose absolute square give the power spectrum

$$\frac{dI(\omega)}{d\omega} = A \frac{1}{(\Gamma/2)^2 + (\omega - \omega_0 - \Delta\omega)^2},$$

which is called the "resonant line shape" or Lorentzian. The total energy radiated is

$$\begin{aligned} I_0 &= A \int_0^\infty d\omega \frac{1}{(\Gamma/2)^2 + (\omega - \omega_0 - \Delta\omega)^2} \\ &= \frac{2A}{\Gamma} \left[\frac{\pi}{2} + \tan^{-1}\left(\frac{2(\omega_0 + \Delta\omega)}{\Gamma} \right) \right] \to \frac{2\pi A}{\Gamma} \end{aligned}$$

where the last expression assumed $\Gamma \ll \omega_0$ as $\omega_0 \tau \ll 1$. This determines $A = I_0 \Gamma / 2\pi$, so the power spectrum is

$$\frac{dI(\omega)}{d\omega} = I_0 \frac{\Gamma}{2\pi} \frac{1}{(\Gamma/2)^2 + (\omega - \omega_0 - \Delta\omega)^2}.$$

In terms of wavelengths, the classical line width is

$$\Delta\lambda = \Gamma |d\lambda/d\omega| = 2\pi c \Gamma/\omega_0^2 = 2\pi c\tau = 1.18 \times 10^{-14} \text{ m}.$$

Quantum mechanically there are oscillator strength factors, but the order of magnitude is correct, so $\Gamma/\omega_0 \sim 10^{-8}$ for optical transitions in atoms, justifying our assumption that $\omega_0 \tau \ll 1$.

Jackson points out that the level shift, classically proportional to $\omega_0^3 \tau^2$, has a coefficient which is not correct quantum mechanically.

Scattering by an Oscillator

A charged oscillator radiates away its energy. It also scatters light. Assume the electron is bound by a spherically symmetric spring with spring constant $m\omega_0^2$. An incoming electric field exerts the force

$$\vec{F}_{\text{ext}} = -m\omega_0^2\vec{x} + e\vec{\epsilon}E_0 e^{i\vec{k}\cdot\vec{x}-i\omega t}.$$

From (1) we have

$$\begin{aligned} m\dot{\vec{v}} &= -m\omega_0^2\vec{x} + e\vec{\epsilon}E_0 e^{i\vec{k}\cdot\vec{x}-i\omega t} \\ &\qquad -\tau m\omega_0^2\dot{\vec{x}} - i\omega\left(\tau - \vec{v}\cdot\vec{k}\right)e\vec{\epsilon}E_0 e^{i\vec{k}\cdot\vec{x}-i\omega t}. \end{aligned}$$

Drop terms proportional to $\vec{v}E_0$ (we didn't consider the magnetic field either) so we have

$$\ddot{\vec{x}} + \Gamma_t\dot{\vec{x}} + \omega_0^2\vec{x} = \frac{eE_0}{m}\vec{\epsilon}(1 - i\omega\tau)e^{i\vec{k}\cdot\vec{x}-i\omega t}.$$

where Γ should be $\tau\omega_0^2$, but we will throw in an additional unspecified damping Γ' due to unspecified "other dissipative processes".

So $\Gamma_t = \tau\omega_0^2 + \Gamma'$. Here we are looking for a steady state solution to this inhomogeneous linear equation, rather than the decay of the homogeneous one, and it is

$$\vec{x}(t) = \frac{eE_0}{m}\,\vec{\epsilon}\,\frac{(1 - i\omega\tau)e^{-i\omega t}}{\omega_0^2 - \omega^2 - i\omega\Gamma_t}.$$

Larmor tells us the power into $d\Omega$ with polarization $\vec{\epsilon}\,'$ is

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{1}{2}\frac{e^2}{4\pi c^3}\left|\vec{\epsilon}\,'\cdot\left(\hat{n}\times(\hat{n}\times\ddot{\vec{x}})\right)\right|^2 = \frac{e^2}{8\pi c^3}\left|\vec{\epsilon}\,'\cdot\ddot{\vec{x}}\right|^2 \\ &= \frac{e^2}{8\pi c^3}\left(\frac{eE_0}{m}\right)^2\left|\frac{(1 - i\omega\tau)\omega^2}{\omega_0^2 - \omega^2 - i\omega\Gamma_t}\right|^2\left|\vec{\epsilon}\,'\cdot\vec{\epsilon}\right|^2.\end{aligned}$$

Dividing by the incoming flux density $cE_0^2/8\pi$, we get the cross section

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{m^2c^4}\frac{(1+\omega^2\tau^2)\omega^4}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma_t^2}\left|\vec{\epsilon}\,'\cdot\vec{\epsilon}\right|^2.$$

We can drop the $\omega^2\tau^2$ compared to 1.

To calculate the total cross section, as for the Thomson cross section, we have $|\vec{\epsilon}' \cdot \vec{\epsilon}|^2 \to 8\pi/3$, so

$$\sigma_T = \frac{8\pi}{3} \frac{e^4}{m^2 c^4} \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma_t^2}.$$

Writing this in terms of the radiation damping width $\Gamma = \omega_0^2 \tau = 2e^2\omega_0^2/3mc^3$ and the resonant wavelength $\lambda := 2\pi c/\omega_0$,

$$\sigma_T = \frac{3}{2\pi} \lambda^2 \frac{\omega^4 \Gamma^2/\omega_0^2}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma_t^2}.$$

At low frequencies we have ω^4 behavior, as predicted by Rayleigh's law, and at high frequencies $\sigma_T \to 6\pi(c\tau)^2 = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2$, the Thomson cross section. This makes sense, in that if the incoming frequency is much higher than the resonant frequency, the electron doesn't realize it is not free. The strong peak at the resonant frequency $\omega = \omega_0$ is called **resonance fluorescence**.